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Engineering Notes

Consensus-Based Cooperative Geometrical Rules for Simultaneous Target Interception

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I. Introduction

IN MODERN warfare, strategic targets can be defended by surface-to-air pursuers, which pose a great threat to attacking pursuers that attempt to destroy the target. One of the effective countermeasures against these defending pursuers is attacking the target simultaneously by multiple pursuers. The simultaneous interception offers the major advantage that even if some of the pursuers are neutralized by the target defenses, the target can still be intercepted by the remaining pursuers. The pursuers are usually modeled with single-integrator, double-integrator, or unicycle kinematics. In our work, we will use pursuers modeled as unicycles, which are vehicles with nonlinear kinematics and can be controlled using angular and translational speeds. Such a model can be used to represent simplified models of pursuers, cars, robots, etc.

The simultaneous interception problem can be solved by guiding the pursuers in the system to appropriate trajectories toward the target. In general, a guidance problem has two levels [1]. The first one is a geometrical rule, which describes the desired kinematics of the engagement. The second one is a guidance law, which is the implementation of the geometrical rule. An example of a geometrical rule is parallel navigation, in which the angle of the line of sight (LOS) between the pursuer and the target is kept constant throughout the engagement, leading the pursuers into a collision triangle. The corresponding guidance law is the well-known proportional navigation guidance (PNG) law, where the lateral acceleration command is proportional to the rate of change of the LOS angle. In this work, we propose cooperative geometrical rules and corresponding guidance laws to achieve the simultaneous interception of a stationary target.

The problem of guiding a single pursuer to a target at a desired impact time has been investigated extensively in the literature. In one of the earliest works, Jeon et al. [2] presented an impact time control guidance law combining the PNG law with feedback on the impact time error to achieve a predefined impact time. Sliding mode control-based guidance laws [3,4] have also been used to control the impact time by suitably designing the sliding surfaces. Saleem and Ratnoo [5] proposed a heading-error-based Lyapunov-based guidance law to control the impact time where the desired impact time was limited by the initial conditions. Controlling the impact time via polynomial look-angle shaping was proposed by Tekin et al. for stationary targets [6] and varying speed targets [7]. Tsalik and Shima [8] proposed yet another look-angle shaping-based circular impact time guidance law to achieve a desired impact time for a stationary or a nonmaneuvering moving target.

Guiding a team of pursuers to coordinate the impact time has also been explored in the literature. Jeon et al. [9] introduced a centralized approach called cooperative proportional navigation, which ensured consensus in the impact time by using a time-varying gain for PNG. Two distributed guidance laws for simultaneous interception were proposed by Zhou and Yang [10], based on different time-to-go estimations, and were proven to achieve simultaneous interception using Lyapunov theory. He et al. [11] proposed a two-stage guidance law where initially a decentralized law guided the pursuers to desired initial conditions, and then PNG was used for target capture.

Simultaneous interception can also be accomplished by leading a team of pursuers to rendezvous at the target, where rendezvous refers to arriving at a consensus only in the positions of the pursuers. In the area of multi-agent systems (MASs), one of the earliest results for unicycles was presented by Marshall et al. [12], where they proposed a feedback control input for rendezvous and provided necessary conditions on the controller gains for the same. Dimarogonas and Kyriakopoulos [13] proposed a discontinuous time-invariant feedback control input for rendezvous in both position and orientation of the pursuers. Using a bearing-based control law, Zheng and Sun [14] lead the pursuers to rendezvous by continuously reducing the perimeter of the polygon whose vertices were marked by the pursuers' positions. Rendezvous problems can also be solved by using consensus protocols, which can be broadly classified into three types: average consensus, max-consensus, and min-consensus. As the names suggest, the consensus is achieved if all of the states of the pursuers reach the average [15], maximum [16], or minimum [17] value of the consensus parameter, respectively. To the best of our knowledge, the rendezvous problem has not yet been solved by using a max-consensus protocol.

The cyclic information exchange topology is explored widely in MASs to implement consensus protocols wherein pursuer i receives information from the $i+1$ modulo n pursuer as shown in Fig. 1. The advantage of this framework is its simplicity and the minimum sensor information required. For the rendezvous problem, the cyclic information exchange topology for pursuers with single-integrator kinematics was studied by Bruckstein et al. [18] and was extended to unicycle kinematics by Marshall et al. [12]. In a recent work, Kumar and Mukherjee [19] proposed a cooperative guidance law for a cyclic information exchange framework to solve the simultaneous target interception problem by a team through achieving consensus in the time-to-go of the pursuers. Yet another information exchange topology used commonly in MASs is the leader–follower framework in which the leader is usually independent of the others, whereas the followers depend on information received from the leader. Jadbabaie et al. [20] proved that the followers converge to the leader as long as all of the members in the group are linked to the leader, though not necessarily in a direct way. An extension to the field of missile guidance for unicycles was proposed by Sun et al. [21], where they designed a cooperative guidance law with feedback linearization for finite-time convergence of the impact times of the followers with that of the leader.

In this work, we consider the simultaneous interception problem of a stationary target by a group of pursuers with unicycle kinematics. The pursuers have different speeds, making the system heterogeneous. Using the tools of cyclic and leader–follower information exchange frameworks from MASs, we propose cooperative geometrical rules and corresponding guidance laws to achieve simultaneous target

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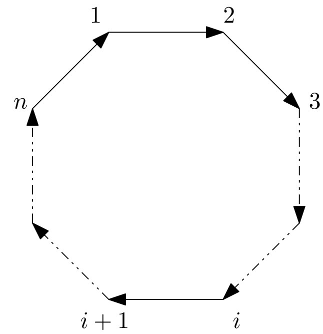


Fig. 1 Cyclic information exchange topology.

interception at the minimum possible time and any desired impact time greater than the minimum, respectively. In the minimum impact time case, we prove that the impact time of each pursuer is guaranteed to converge to the maximum impact time among the pursuers in the case that they are heading straight to the target, resulting in max-consensus. To the best of our knowledge, this is the first work that uses a max-consensus protocol to propose cooperative geometrical rules that lead to simultaneous target interception. Motivated by the work in [8], the implementation of the protocol is based on the circular impact time guidance concept. For the case of desired impact time, the leader imposes the desired impact time independent of the other pursuers in the system, and the followers modify their impact times according to that of the leader, eventually leading to the simultaneous interception. Based on the cyclic and leader–follower frameworks, we present the geometrical rules and show that, by using them, we achieve simultaneous interception from arbitrary initial positions of the pursuers. The corresponding guidance laws are implemented using a Proportional Integral controller. The remainder of the paper is outlined as follows. In Sec. II, the problem is formulated mathematically and the main objective of the paper is presented. The cooperative geometrical rules based on the cyclic and leader–follower frameworks, including their analysis and corresponding guidance laws, are presented in Sec. III. Simulation results are presented in Sec. IV, and concluding remarks are summarized in Sec. V.

II. Problem Formulation

We address the problem of simultaneous target interception by a group of n pursuers in finite time. The pursuers are modeled as unicycles and have constant linear speeds. The target is assumed to be stationary and represents any point of interest like a landmark or a beacon. In Fig. 2, we present a schematic view of the planar geometry between the target and the pursuers. The X_I–O_I–Y_I axes form a Cartesian inertial reference frame. The position coordinates of the target T are given by $(x_T, y_T) \in \mathbb{R}^2$. For every $i \in \{1, 2, \ldots, n\}$, the ith pursuer is denoted by P_i and its position coordinates by $(x_i, y_i) \in \mathbb{R}^2$. Its linear speed, heading angle, and normal acceleration are written as v_i, γ_i, and a_i, respectively. The equations of motion of pursuer P_i are $\dot{x}_i = v_i \cos\gamma_i$, $\dot{y}_i = v_i \sin\gamma_i$, and $\dot{\gamma}_i = a_i/v_i$. The pursuers are assumed to be heterogeneous in terms of their speed, which can be different for all pursuers. The distance and the LOS angle between P_i and the target are denoted by r_i and λ_i, respectively. The look angle, denoted by ϵ_i, is defined as the angle between the velocity vector of P_i and its corresponding LOS to the target. The equations of motion of P_i can also be expressed in a polar coordinate frame attached to the pursuer, as follows:

$$\begin{aligned} \dot{r}_i &= -v_i \cos\epsilon_i \\ \dot{\lambda}_i &= -\frac{v_i \sin\epsilon_i}{r_i} \\ \dot{\gamma}_i &= \frac{a_i}{v_i} \end{aligned} \tag{1}$$

The main objective of this paper is to propose cooperative geometrical rules and corresponding guidance laws that lead to simultaneous target interception by the pursuers. In the next section, we explore some concepts from MASs to achieve this goal.

III. Consensus Protocols for Simultaneous Target Interception

In this section, we present geometrical rules that lead to the simultaneous interception of a stationary target by a group of n heterogeneous pursuers. Our goal is to control the impact time of the pursuers by using consensus protocols. The consensus in the impact times of the pursuers will result in simultaneous target interception. We propose two kinds of geometrical rules that drive the system toward the target while achieving either the minimum possible impact time or any desired impact time that is greater than the minimum possible time.

A. Cyclic Strategy to Achieve a Minimum Impact Time

Here, we present a geometrical rule that leads to simultaneous interception of the target at the minimum possible time. Because simultaneous interception implies the synchronized impact times of the pursuers, we choose the impact time as the consensus parameter. Once consensus is achieved, simultaneous target interception is guaranteed. So, let us define the following parameter:

$$\tilde{t}_i(t) = \frac{r_i(t)}{v_i} \tag{2}$$

which is the time required for P_i to reach the target if it follows a straight-line trajectory toward it. Note that $\tilde{t}_i$ is not the impact time of P_i, but merely the minimum possible value of the impact time of P_i at any given time t. We choose $\tilde{t}_i$ as the consensus parameter because synchronizing this parameter implies simultaneous interception in minimum possible time.

We consider a cyclic information exchange topology among the pursuers, so the geometrical rule for P_i is based on the information from P_{i+1} only. To achieve consensus, we propose that each pursuer compares its $\tilde{t}_i$ with that of its neighbor, and adjusts its trajectory accordingly to synchronize its impact time with that of its neighbor. Furthermore, we choose to adjust the pursuers' $\tilde{t}$ by using circular trajectories because their impact times can be easily controlled in this case, as shown by Tsalik and Shima [8].

For the circular trajectory shown in Fig. 3, the radius can be expressed as $R_i = (r_i/2\sin\epsilon_i)$. Then, the arc length of the circle

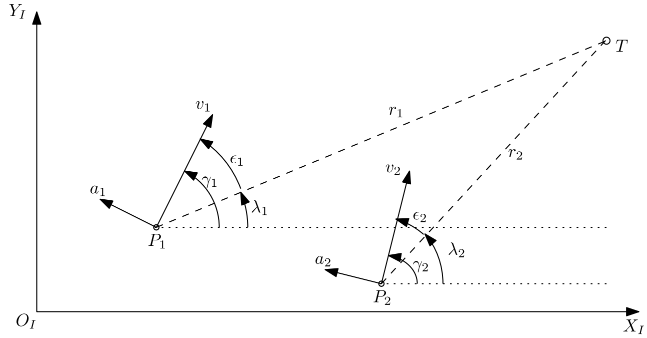


Fig. 2 Planar engagement.

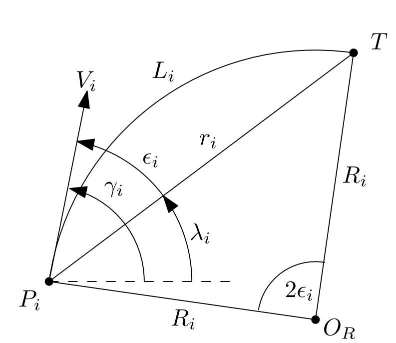


Fig. 3 Circular trajectory geometry.

between P_i and the target is $L_i = 2\epsilon_i R_i = r_i \epsilon_i / \sin\epsilon_i$. As the linear speed is constant, $L_i = v_i \Delta t_i$, where Δt_i is the time in which P_i will reach the target. By combining these expressions, it directly follows that

$$\epsilon_i = \mathrm{Asinc}\left(\frac{r_i}{v_i \Delta t_i}\right) = \mathrm{Asinc}\left(\frac{\tilde{t}_i}{\Delta t_i}\right) \tag{3}$$

where the Asinc function is the inverse function of the sinc function, defined as $\mathrm{sinc}(\epsilon) = [\sin(\epsilon)/\epsilon]$. By controlling ϵ_i, Δt_i can be directly controlled and vice versa.

Now, in the present cyclic information exchange framework, each pursuer follows a circular trajectory while trying to synchronize its impact time with that of its neighbor. As previously discussed, the minimum impact time for P_i is $\tilde{t}_i$. Because we want the simultaneous target interception in minimum time, our goal is to achieve consensus in $\tilde{t}_i$ for every $i \in \{1, 2, \ldots, n\}$. For the same, every P_i modifies its trajectory based on the relation between $\tilde{t}_i$ and $\tilde{t}_{i+1}$ as explained in the following:

1) $\tilde{t}_i < \tilde{t}_{i+1}$: In this case, we set the desired impact time of P_i to the time that its neighbor requires to arrive at the target in a straight line, implying that $\Delta t_i = \tilde{t}_{i+1}$. Using it in Eq. (3), we get the suitable ϵ_i. Then, P_i modifies its circular trajectory to have a longer arc to the target. This increases the time to reach the target while synchronizing its impact time with that of P_{i+1}.

2) $\tilde{t}_i \geq \tilde{t}_{i+1}$: Here, P_i tries to reach the target as fast as it can. Hence, it nullifies its look angle and heads straight to the target.

Based on this logic, the geometrical rule can be expressed mathematically as follows:

$$\epsilon_i(t) = \begin{cases} \mathrm{Asinc}\left(\dfrac{\tilde{t}_i(t)}{\tilde{t}_{i+1}(t)}\right) & \tilde{t}_i(t) < \tilde{t}_{i+1}(t) \\ 0 & \tilde{t}_i(t) \geq \tilde{t}_{i+1}(t) \end{cases} \tag{4}$$

By following this geometrical rule, P_i adjusts its trajectory according to its neighbor, when $\tilde{t}_i(t) < \tilde{t}_{i+1}(t)$. Therefore, the impact time $\tilde{t}_i$ of P_i eventually converges to $\max_{i\in\mathbb{N}_n}(\tilde{t}_i)$ of the group, leading to max-consensus. Once consensus is achieved, all of the pursuers synchronize their impact times and reach the target simultaneously. Next, we will prove that, for the proposed geometrical rule, the system converges to $\max_{i\in\mathbb{N}_n}(\tilde{t}_i)$.

Theorem 1: Consider n pursuers with unicycle kinematics in a cyclic information exchange framework. The trajectories of the pursuers are governed by the geometrical rule described in Eq. (4). Simultaneous interception of any predetermined target point is guaranteed from arbitrary initial positions of the pursuers by achieving max-consensus in their impact times at the target. Moreover, interception is achieved at the minimum possible time, which is equal to the maximum impact time among the pursuers in the case that they are heading straight to the target.

Proof: Without any loss of generality, let us assume that $\tilde{t}_1(0) \neq \tilde{t}_2(0) \neq \ldots \neq \tilde{t}_n(0)$. Then, we denote P_k as the unique pursuer that satisfies the condition $\tilde{t}_k(0) = \max_{i\in\mathbb{N}_n}[\tilde{t}_i(0)]$. This leads us to define a difference function for every P_i as

$$d_i(t) \equiv \tilde{t}_k(t) - \tilde{t}_i(t) \tag{5}$$

We first prove that $\tilde{t}_k(t)$ remains the maximum value for all $t > 0$, equivalently $\tilde{t}_k(t) = \max_{i\in\mathbb{N}_n}[\tilde{t}_i(t)]\forall t$. Assume that for some $t^* > 0$, one of the pursuers in the system, denoted as P_j, has a greater $\tilde{t}$ than P_k, leading to $d_j(t^*) < 0$. Now, for every i, $\dot{d}_i(t) = -\cos\epsilon_k(t) + \cos\epsilon_i(t)$ is bounded, implying that $d_i(t)$ is continuous [22]. Applying the intermediate value theorem, there exists a time $t^\dagger < t^*$ at which $d_j(t^\dagger) = 0$ and $\tilde{t}_j(t^\dagger) = \tilde{t}_k(t^\dagger)$. According to Eq. (4), $\epsilon_j(t^\dagger) = \epsilon_k(t^\dagger) = 0$, meaning that for $t = t^\dagger$ both pursuers move in a straight line toward the target. Now, $\tilde{t}_j$ will be greater than $\tilde{t}_k$ only if P_j stops moving in a straight line toward the target. According to Eq. (4), this will occur only if $\tilde{t}_j(t) < \tilde{t}_{j+1}(t)$ for $t > t^\dagger$. Because of continuity, this implies that there exists a time t at which $\tilde{t}_{j+1}(t) = \tilde{t}_j(t)$, meaning that P_{j+1} will also move in a straight line to the target and its $\tilde{t}$ will not be greater than $\tilde{t}_j$. Following the same line of thought for all the pursuers in the system leads to the contradiction of the assumption that t^* exists. Therefore, we derive that for every pursuer $d_i(t) \geq 0\forall t$ and $\tilde{t}_k(t) = \max_{i\in\mathbb{N}_n}[\tilde{t}_i(t)]\forall t$.

Now, we will show that target interception occurs simultaneously at $t = \tilde{t}_k(0)$. We already know that $\tilde{t}_k(t) = \max_{i\in\mathbb{N}_n}[\tilde{t}_i(t)]\forall t$. Using this condition in Eq. (4), we get $\epsilon_k(t) = 0\forall t$, which means that P_k moves in a straight line to the target. Therefore, $\tilde{t}_k(t) = \tilde{t}_k(0) - t$ and P_k intercepts the target at $t = \tilde{t}_k(0)$. Since $\tilde{t}_k(t) = \max_{i\in\mathbb{N}_n}[\tilde{t}_i(t)]\forall t$, we conclude that $\tilde{t}_i(t = \tilde{t}_k(0)) = 0\forall i$. Now, let us assume that P_{k-1} intercepts the target at $t^* < \tilde{t}_k(0)$, then $\tilde{t}_{k-1}(t^*) = 0$ and $r_{k-1}(t^*) = 0$. Since $\tilde{t}_k(t^*) > 0$, Eq. (4) gives $\epsilon_{k-1}(t^*) = \mathrm{Asinc}(0) = \pi$. Since $\epsilon_{k-1}(t)$ is a continuous function, there exists δ_t, in which

$$\frac{\pi}{2} < \epsilon_{k-1}(t) < \frac{3\pi}{2} \quad \forall\ t \in (t^* - \delta_t, t^*]$$

This implies that $\dot{r}_{k-1}(t) > 0$ in this interval; hence P_{k-1} moves away from the target. This contradicts the assumption of existence of $t^* < \tilde{t}_k(0)$ at which $r_{k-1}(t^*) = 0$. By applying this analysis for the rest of the pursuers in the system, it follows directly that all of them intercept the target simultaneously at $t = \tilde{t}_k(0)$ as desired. The assumption of distinct values of $\tilde{t}_i(0)$ was for simplicity, and the proof can be easily extended to cases where the values of $\tilde{t}_i(0)$ are not distinct. □

In conclusion, the impact time of each pursuer converges to the impact time of the pursuer with the maximum value of the initial $\tilde{t}_i$, thereby leading to max-consensus. It is also important to note that convergence is guaranteed for all possible initial positions of the pursuers.

Corollary 1: Given a pursuer P_k that heads straight to the target, the trajectory of pursuer P_{k-1} is circular.

Proof: As mentioned in the proof of Theorem 1, $\tilde{t}_k(t) = \max_{i\in\mathbb{N}_n}[\tilde{t}_i(t)]\forall t \in [0, \tilde{t}_k(0)]$, which leads to $\tilde{t}_{k-1}(t) \leq \tilde{t}_k(t)\forall t \in [0, \tilde{t}_k(0)]$. Using Eq. (4), we obtain

$$\epsilon_{k-1}(t) = \mathrm{Asinc}\left(\frac{\tilde{t}_{k-1}(t)}{\tilde{t}_k(t)}\right) = \mathrm{Asinc}\left(\frac{\tilde{t}_{k-1}(t)}{\tilde{t}_k(0) - t}\right) \tag{6}$$

By using the work presented in [8], we know that Eq. (6) represents a geometrical rule that leads to a circular trajectory. Hence, the trajectory of pursuer P_{k-1} is circular. □

Remark 1: Without any loss of generality, we have chosen positive values of the look angle for the geometrical rule. This choice enforces the pursuers' trajectory direction to be clockwise. By changing the sign of the look angle, the direction of the trajectories will become counterclockwise, and the results presented so far will hold for negative values of look angle as well.

Having proved the convergence properties of the geometrical rule given in Eq. (4), we will now propose a geometrical rule for imposing a desired impact time.

B. Leader–Follower Strategy to Achieve a Desired Impact Time

Adjusting the impact time, and particularly increasing it, can be useful in cases where we want a longer engagement. In this section, we propose a leader–follower strategy to achieve a desired impact time, which is bounded below by the minimum possible impact time. We choose one of the pursuers as the leader, denoted by P_0, and force it to impose a circular trajectory with a given desired time independent of the other pursuers in the system. The remaining pursuers in the system, called the followers, communicate with the group using either an n-to-one or a one-to-one communication topology, which are defined below.

Definition 1: The n-to-one communication topology: In this case, for $i \in \{1, \ldots, n-1\}$, every pursuer P_i modifies its trajectory with

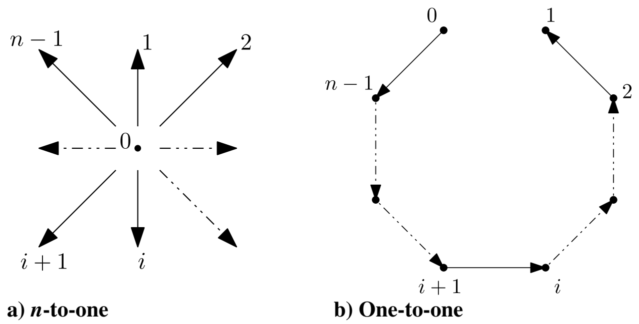


Fig. 4 Leader–follower communication topology.

respect to the information transferred from the leader P_0 as shown in Fig. 4a.

Definition 2: The one-to-one communication topology: Here, for $i \in \{1, \ldots, n-1\}$, every pursuer P_i modifies its trajectory with respect to the information transferred from pursuer P_{i+1}, except pursuer P_{n-1}, which receives its information from the leader P_0. The information flow is depicted in Fig. 4b.

The choice between the two communication topologies depends on the sensing capabilities of the system. The one-to-one topology should be used when the leader cannot communicate with all of the pursuers due to sensing limitations, and the followers can communicate with each other. The n-to-one topology should be used in cases where the leader can transmit information to all the pursuers.

Now, we propose the geometrical rule for each of the two communication topologies. Note that, in both cases, we can choose P_0 arbitrarily as any one of the pursuers, where the choice must be made a priori and cannot be changed during the engagement.

For the n-to-one topology, the leader modifies its impact time by elongating its trajectory with respect to the desired impact time. This is done by using Eq. (3) for the leader P_0 and replacing Δt_i with $t_d - t$, where t_d is the desired impact time. Note that $t_d - t$ represents the time-to-go with respect to the desired impact time. The followers modify their impact time by elongating their trajectories with respect to the leader, which is achieved by replacing Δt_i with $\tilde{t}_0$ in Eq. (3). The geometrical rule can then be expressed mathematically as follows:

$$\epsilon_0(t) = \mathrm{Asinc}\left(\frac{\tilde{t}_0(t)}{t_d - t}\right)$$
$$\epsilon_i(t) = \begin{cases} \mathrm{Asinc}\left(\dfrac{\tilde{t}_i(t)}{\tilde{t}_0(t)}\right) & \tilde{t}_i(t) < \tilde{t}_0(t) \\ 0 & \tilde{t}_i(t) \geq \tilde{t}_0(t) \end{cases} \quad i = 1, 2, \ldots, n-1 \tag{7}$$

For the one-to-one topology, the leader modifies its impact time in the same way described for the n-to-one communication topology. However, instead of referring to the leader P_0, the followers refer to their neighbors. So, each follower P_i modifies its impact time by elongating its trajectory with respect to pursuer P_{i+1} by replacing Δt_i with $\tilde{t}_{i+1}$ in Eq. (3). The geometrical rule can then be expressed mathematically as

$$\epsilon_0(t) = \mathrm{Asinc}\left(\frac{\tilde{t}_0(t)}{t_d - t}\right)$$
$$\epsilon_i(t) = \begin{cases} \mathrm{Asinc}\left(\dfrac{\tilde{t}_i(t)}{\tilde{t}_{i+1}(t)}\right) & \tilde{t}_i(t) < \tilde{t}_{i+1}(t) \\ 0 & \tilde{t}_i(t) \geq \tilde{t}_{i+1}(t) \end{cases} \quad i = 1, 2, \ldots, n-1 \tag{8}$$

Note that t_d is greater than the minimum possible impact time. Next, we will prove that by using the proposed geometrical rules, simultaneous interception is achieved at the desired impact time.

Theorem 2: Consider n pursuers modeled with unicycle kinematics and located at arbitrary initial positions, attempting to intercept a stationary target at a desired impact time t_d that satisfies the following condition:

$$t_d > \max_{i \in \mathbb{N}_n} [\tilde{t}_i(0)]$$

Simultaneous interception of the target at time t_d is always guaranteed by following the geometrical rule given in either Eq. (7) or Eq. (8), where the underlying communication topology between the pursuers is either n-to-one or one-to-one, respectively.

Proof: To begin with, we consider the n-to-one leader–follower communication topology. As mentioned before, the leader P_0 can be chosen arbitrarily as any one of the pursuers. Let us define a virtual pursuer P_v such that $\tilde{t}_v(0) = t_d > \max_{i \in \mathbb{N}_n}[\tilde{t}_i(0)]$. Let us consider any follower P_i, where $i \in \{1, 2, \ldots, n-1\}$. In an n-to-one topology, every P_i receives information solely from P_0, irrespective of the other pursuers in the system. Now, consider the subsystem consisting of pursuer P_i, leader P_0, and the virtual pursuer P_v in a cyclic information exchange framework. Here, P_i receives information from P_0, P_0 from P_v, and P_v from P_i, while following the geometrical rule in Eq. (4).

From the proof of Theorem 1, we know that the pursuer that heads straight to the target initially continues on a straight trajectory. So, P_v moves in a straight line as $\tilde{t}_v(t) = \max\{\tilde{t}_i(t), \tilde{t}_0(t), \tilde{t}_v(t)\} \forall t \in [0, t_d]$. For the leader P_0, we get $\Delta t_0(t) = \tilde{t}_v(t) = t_d - t$ as $\tilde{t}_0(t) < \tilde{t}_v(t)$, and P_0 follows a circular trajectory according to Corollary 1. As P_i adjusts its trajectory with respect to P_0, we get $\Delta t_i = \tilde{t}_0$. Substituting these expressions in Eq. (4), we get the geometrical rule given in Eq. (7). Now, from Theorem 1, we know that the geometrical rule leads to simultaneous target interception by the subsystem in time t_d. Because the choice of i is arbitrary, the proof holds for every $i \in \{1, 2, \ldots, n-1\}$.

For the one-to-one case, we also introduce the virtual pursuer P_v, which is defined exactly as in the previous case. Now, we consider the system consisting of the $n-1$ followers, the leader P_0, and the virtual pursuer P_v in a cyclic information exchange framework. The rest of the analysis is the same as the n-to-one case, which eventually guarantees simultaneous target interception at the desired time t_d under the geometrical rule given in Eq. (8). □

To conclude, we have proven that simultaneous interception at a desired impact time can be achieved in two different communication topologies by using the appropriate geometrical rule. Moreover, the interception is guaranteed from any initial position of the pursuers, as long as the desired impact time is larger than $\max_{i \in \mathbb{N}_n}[\tilde{t}_i(0)]$. After deriving the geometrical rules, we will now design corresponding guidance laws and investigate their ability to provide simultaneous interception in the presence of heading errors.

C. Guidance Laws Design

In this subsection, we design simple guidance laws to enforce the geometrical rules defined in the previous subsection. By enforcing the geometrical rule, we dictate the desired look angle for each pursuer in the system. In this work, we propose to control the look angle with a PI controller, as designed in [8,23]. The goal of the guidance laws is to eliminate the look-angle deviations from the nominal trajectories, which correspond to the specified impact times. The look-angle error $\Delta\epsilon_i$ is obtained as the difference between the desired look angle, denoted by ϵ_i^d, and the current look angle ϵ_i. The value of ϵ_i^d is obtained from the geometrical rule, whereas ϵ_i is calculated from the kinematics equations. The error in the look angle is passed to a PI controller to obtain the control input

$$a_i = K_1 \Delta\epsilon_i + \frac{K_2 \Delta\epsilon_i}{s} \tag{9}$$

where K_1 is the proportional gain, and K_2 is the integral gain. Note that the value of ϵ_i^d is different for each of the geometrical rules obtained in the previous subsection. Therefore, we obtain a different guidance law for each geometrical rule. Using the kinematics shown in Eq. (1), we derive γ_i. In the next step, we calculate the LOS angle by

$$\lambda_i = \arctan\left(\frac{y_t - y_i}{x_t - x_i}\right)$$

to obtain $\epsilon_i = \gamma_i - \lambda_i$.

The proposed controller can mitigate heading errors using suitable controller gains K_1 and K_2. The generated lateral acceleration commands synchronize the $\tilde{t}_i$ of every pursuer P_i with that of its neighbor by eliminating the look-angle error and does not focus on driving the trajectories directly to the nominal one. Therefore, the pursuers can achieve consensus in $\tilde{t}$s, and consequently simultaneous interception, but the impact time can deviate from its specified value.

IV. Simulations Results

The theoretical results obtained so far are reinforced in this section via numerical simulations. First, we examine the cooperative geometrical rules in ideal scenarios for the different communication topologies discussed before. Next, we analyze the proposed guidance laws by applying initial heading errors to the pursuers. Throughout this section, the target is stationary and is located at the origin. In all figures, the target and the pursuers' launch points are denoted by an asterisk and a square box, respectively.

A. Simulation of Geometrical Rules

In this subsection, we present the working principles of the geometrical rules. The pursuers' look angles are obtained directly from one of the geometrical rules in Eq. (4) or Eq. (7). No additional physical constraints are included in this section.

1. *Minimum Impact Time Geometrical Rule*

Here, we investigate the minimum impact time geometrical rule described in Eq. (4). The scenario is ideal, implying that no heading errors are present and the kinematics of the pursuers are governed by the desired look angles obtained from the geometrical rule. The pursuers are launched from $(x_i, y_i) = (-7071\ \mathrm{m}, -7071\ \mathrm{m})$ for every $i \in \{1, \ldots, 4\}$. They approach the target at different speeds given by $v_1 = 230\ \mathrm{m/s}$, $v_2 = 220\ \mathrm{m/s}$, $v_3 = 210\ \mathrm{m/s}$, and $v_4 = 200\ \mathrm{m/s}$. The information exchange topology between the pursuers is cyclic.

The resulting trajectory, the parameter $\tilde{t} = r/v$, the look angle, and the lateral acceleration are shown in Fig. 5. As illustrated in the figure, the pursuers reach the target simultaneously at $t = 50$ s. We have

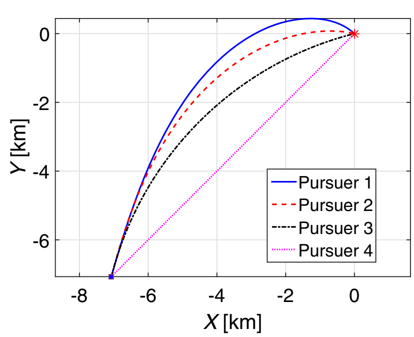
$$\tilde{t}_4(0) = \frac{r_4(0)}{v_4} = \max_{i\in\mathbb{N}_n}[\tilde{t}_i(0)] = 50\ \mathrm{s}$$

which validates the claim in Theorem 1 that the common impact time is $\max_{i\in\mathbb{N}_n}[\tilde{t}_i(0)]$. Also, Fig. 5b shows that $\tilde{t}_4(t) = \max_{i\in\mathbb{N}_n}[\tilde{t}_i(t)] \forall t \in [0, 50]$ which is consistent with the theoretical results. Hence, the trajectory of P_4 is a straight line, as depicted in Fig. 5a, and ϵ_4 remains zero throughout the engagement, as shown in Fig. 5d. According to Fig. 5b, $\tilde{t}_3(t) < \tilde{t}_4(t)$, $\tilde{t}_2(t) < \tilde{t}_3(t)$, and $\tilde{t}_1(t) < \tilde{t}_2(t) \forall t$. Then, by Eq. (4), pursuers P_1, P_2, and P_3 do not move in a straight line toward the target throughout the engagement, which is seen in Fig. 5d.

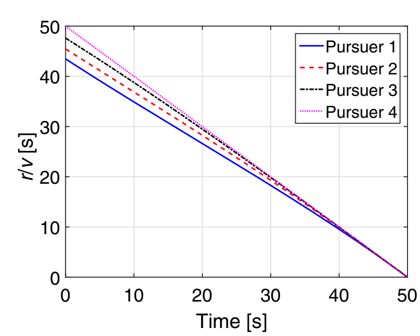
In Fig. 5c, we see that the lateral acceleration of P_4 is zero throughout the engagement owing to its straight-line trajectory. So, the trajectory of P_3 is circular (Fig. 5a) as claimed in Corollary 1, and ϵ_3 profile is linear with respect to time (Fig. 5d). Additionally, the lateral acceleration of P_3 is constant, due to its circular trajectory. P_1 and P_2 continually update their instantaneous circular trajectories as they receive information from pursuers that do not remain on straight lines, making their trajectories noncircular. The terminal lateral acceleration of P_1 is high (Fig. 5c), which could degrade the performance when the acceleration is limited. This situation occurs as the terminal value of $\dot{\epsilon}$ is very high for P_1. We have also observed that the terminal lateral acceleration demand increases as the number of pursuers increases.

2. *Desired Impact Time Geometrical Rule*

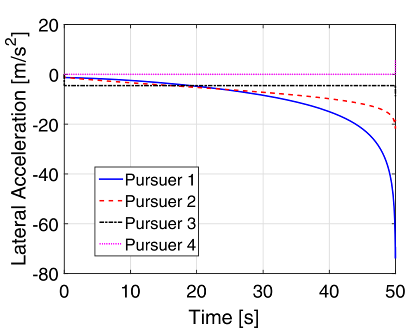
In this part, we examine the desired impact time geometrical rule defined in Eq. (7) for the n-to-one communication topology. Again, the scenario is ideal and the pursuers are launched from $(x_i, y_i) = (-7071\ \mathrm{m}, -7071\ \mathrm{m})$ for every $i \in \{1, \ldots, 4\}$. Their speeds are $v_1 = 200\ \mathrm{m/s}$, $v_2 = 240\ \mathrm{m/s}$, $v_3 = 230\ \mathrm{m/s}$, and $v_4 = 210\ \mathrm{m/s}$.



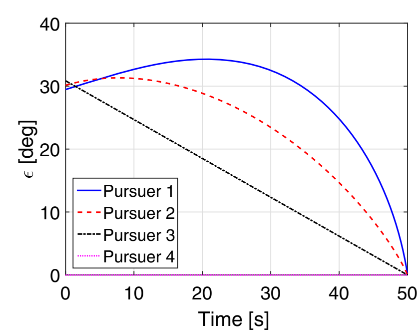
a) Planar trajectories



b) Distance to speed ratio - $\tilde{t}$

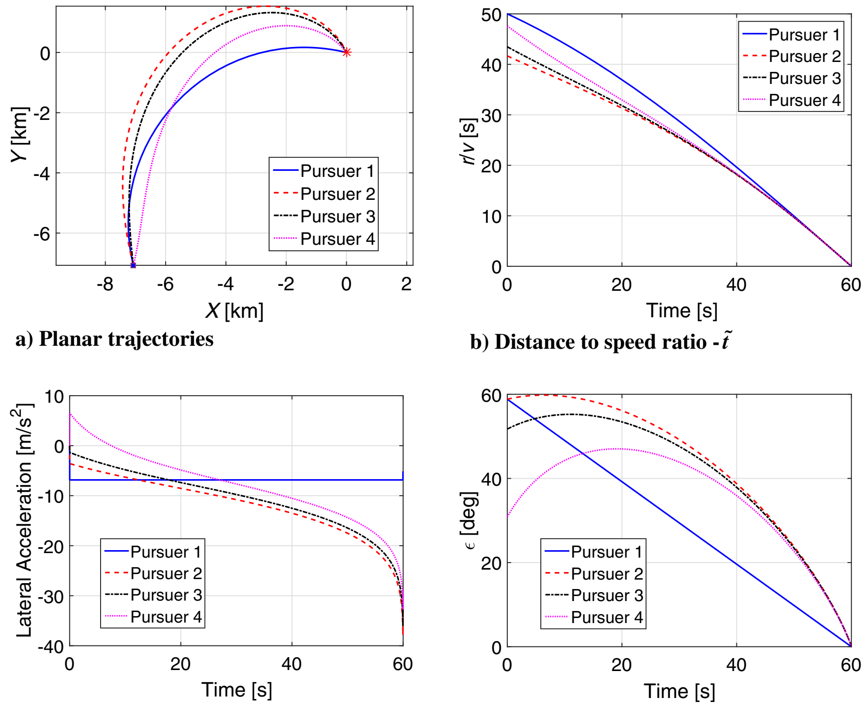


c) Lateral acceleration



d) Look angle

Fig. 5 Simulation results of minimum impact time geometrical rule.



a) Planar trajectories

b) Distance to speed ratio - $\tilde{t}$

c) Lateral acceleration

d) Look angle

Fig. 6 Simulation results of desired impact time n-to-one geometrical rule.

$$t_d = 60 \text{ s} > \max_{i=1,2,\ldots n}[\tilde{t}_i(0)] = \frac{r_1(0)}{v_1} = 50 \text{ s}$$

Pursuer P_1 is defined as the leader and the rest of the pursuers use only its information.

The results in Fig. 6 show that the pursuers achieve simultaneous interception at the desired impact time. The trajectory of P_1 (Fig. 6a) is circular, as depicted by ϵ_1 profile in Fig. 6d. According to Fig. 6b, $\tilde{t}_{2,3,4}(t) < \tilde{t}_1(t)$ throughout the engagement. Then, by Eq. (7), none of the pursuers move in a straight line toward the target throughout the engagement, which is shown in Fig. 6d.

B. Simulation of Guidance Laws

In this subsection, the proposed PI-controller-based guidance law is analyzed with controller gains $K_1 = 800 \text{ m}/(\text{rad} \cdot \text{s}^2)$ and $K_2 = 30 \text{ m}/(\text{rad} \cdot \text{s}^3)$. For the controller, the desired look angle is obtained by one of the geometrical rules in Eq. (4) or Eq. (8). The look-angle error is fed to the controller to calculate lateral acceleration commands, which are bounded by the maximum value of 100 m/s^2. Using these commands, we generate the pursuers' trajectories.

1. *Minimum Impact Time Guidance Law*

Here, the minimum impact time guidance law is analyzed for pursuers following a cyclic information exchange topology. The launch points of the pursuers are $(x_1, y_1) = (-7071 \text{ m}, -7071 \text{ m})$, $(x_2, y_2) = (10{,}000 \text{ m}, 9000 \text{ m})$, $(x_3, y_3) = (5000 \text{ m}, -6000 \text{ m})$, and $(x_4, y_4) = (-8000 \text{ m}, 0 \text{ m})$. The pursuers' speeds are $v_1 = 125 \text{ m/s}$, $v_2 = 180 \text{ m/s}$, $v_3 = 170 \text{ m/s}$, and $v_4 = 150 \text{ m/s}$. Additionally, we introduce heading errors $\gamma_{1_{\text{err}}} = 180°$, $\gamma_{2_{\text{err}}} = -10°$, $\gamma_{3_{\text{err}}} = 150°$, and $\gamma_{4_{\text{err}}} = 20°$. The results in Fig. 7 show that, despite the large heading errors, the interception of the target occurs simultaneously. However, the interception occurs at $t = 84$ s instead of

$$\max_{i \in \mathbb{N}_n}[\tilde{t}_i(0)] = \tilde{t}_1(0) = \frac{r_1(0)}{v_1} = 80 \text{ s}$$

Figure 7a shows that P_1's trajectory tends to be a straight line, and its $\tilde{t}$ stays the maximum throughout the engagement (Fig. 7b). P_4 communicates with P_1 and tends to move in a circular trajectory. Now, the trajectory of P_3 is nonstraight as $\tilde{t}_3 \le \tilde{t}_4 \forall t$. Initially, P_2 moves in a straight line as $\tilde{t}_2 < \tilde{t}_3$. After 50 s, $\tilde{t}_2 > \tilde{t}_3$ and its trajectory becomes nonstraight. Figure 7d shows that, in the first 5 s of the engagement, the pursuers reach their lateral acceleration limiter due to large initial heading errors. By $t = 10$ s the lateral acceleration of each pursuer approaches a nominal value. At $t = 50$ s, P_2 changes its trajectory from a straight line to a nonstraight line, which leads to a spike in its lateral acceleration. From Fig. 7c, it should be noted that the controller does not eliminate the look-angle errors perfectly, which leads to high lateral accelerations in the last second of the engagement. Nevertheless, simultaneous interception of the target still occurs.

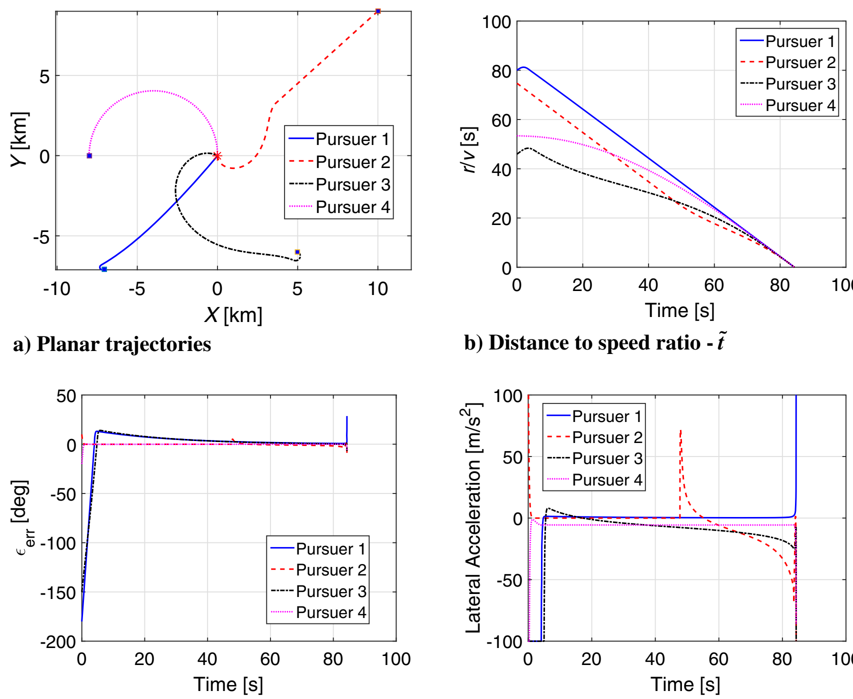
2. *Desired Impact Time Guidance Law*

Now, we examine the desired impact time guidance law. The pursuers follow a one-to-one communication topology, and the desired look angles are obtained from the desired impact time geometrical rule in Eq. (8). They are launched from $(x_1, y_1) = (-7071 \text{ m}, -7071 \text{ m})$, $(x_2, y_2) = (0 \text{ m}, 12{,}000 \text{ m})$, $(x_3, y_3) = (8000 \text{ m}, -4000 \text{ m})$, and $(x_4, y_4) = (-5000 \text{ m}, 0 \text{ m})$. Their speeds are $v_1 = 210 \text{ m/s}$, $v_2 = 180 \text{ m/s}$, $v_3 = 220 \text{ m/s}$, and $v_4 = 240 \text{ m/s}$. There are also heading errors $\gamma_{1_{\text{err}}} = 120°$, $\gamma_{2_{\text{err}}} = -40°$, $\gamma_{3_{\text{err}}} = -120°$, and $\gamma_{4_{\text{err}}} = 45°$. We have

$$t_d = 70 \text{ s} > \max_{i=1,2,\ldots n}[\tilde{t}_i(0)] = \frac{r_2(0)}{v_2} = 66.6 \text{ s}$$

Pursuer P_1 is defined as the leader. Pursuer P_2 uses information from P_3, P_3 from P_4, and P_4 from the leader P_1.

Figure 8 shows that simultaneous interception is achieved at the desired impact time despite the heading errors. As depicted in Fig. 8a, the initial trajectory of P_1 is not circular due to its heading error. The trajectories of P_2 and P_3 are straight for a limited time interval. Their trajectories change due to the relative changes in their $\tilde{t}$s as shown in Fig. 8b. The lateral accelerations of the pursuers are presented in Fig. 8d. Initially, the pursuers are required to overcome the heading errors and reach their maximum allowed lateral acceleration in the



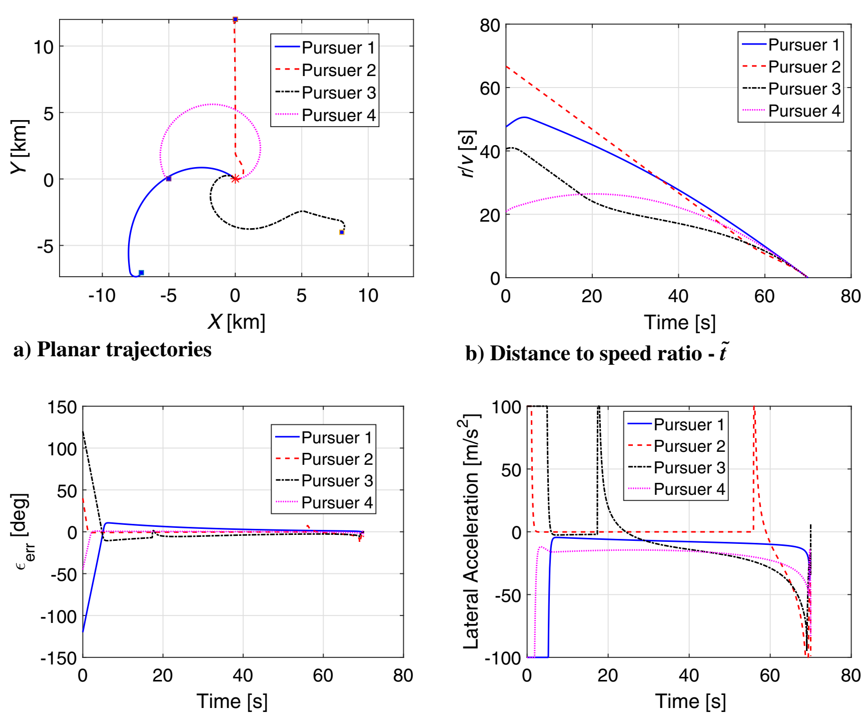
a) Planar trajectories

b) Distance to speed ratio - $\tilde{t}$

c) Look angle error

d) Lateral acceleration

Fig. 7 Simulation results of minimum impact time guidance law.



a) Planar trajectories

b) Distance to speed ratio - $\tilde{t}$

c) Look angle error

d) Lateral acceleration

Fig. 8 Simulation results of desired impact time one-to-one guidance law.

process. Moreover, even though the look-angle errors (Fig. 8c) are not eliminated perfectly, simultaneous interception still occurs.

V. Conclusions

Cooperative geometrical rules and corresponding guidance laws to achieve simultaneous target interception for a team of n pursuers were presented and investigated. To achieve simultaneous interception at the minimum possible time, a cyclic information exchange framework was applied, where each pursuer modified its trajectory with respect to its neighbor. Max-consensus occurs in the impact times of the group, where the maximum corresponds to the largest impact time among the pursuers in the case that they are heading straight to the target. To achieve simultaneous interception at a desired impact time, the underlying geometrical rule was implemented using a leader–follower framework. The leader imposed the desired impact time, and the followers modified their trajectories according to it. Two communication topologies were examined in the leader–follower framework: the one-to-one topology and the n-to-one topology. For these communication topologies, it has been proved that simultaneous interception is achieved at the desired impact time. It is also established that simultaneous interception is guaranteed from arbitrary initial conditions for both minimum and desired impact time geometrical rules. The effectiveness of the guidance laws was shown by achieving simultaneous interception in the presence of large heading errors.

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References

[1] Shneydor, N. A., *Missile Guidance and Pursuit: Kinematics, Dynamics and Control*, Woodhead Publ., Cambridge, England, U.K., 1998, pp. 1–10.

[2] Jeon, I.-S., Lee, J.-I., and Tahk, M.-J., "Impact-Time-Control Guidance Law for Anti-Ship Missiles," *IEEE Transactions on Control Systems Technology*, Vol. 14, No. 2, 2006, pp. 260–266.
https://doi.org/10.1109/TCST.2005.863655

[3] Kumar, S. R., and Ghose, D., "Sliding Mode Control Based Guidance Law with Impact Time Constraints," *2013 American Control Conference*, Inst. of Electrical and Electronics Engineers, New York, 2013, pp. 5760–5765.
https://doi.org/10.1109/ACC.2013.6580740

[4] Cho, D., Kim, H. J., and Tahk, M.-J., "Nonsingular Sliding Mode Guidance for Impact Time Control," *Journal of Guidance, Control, and Dynamics*, Vol. 39, No. 1, 2016, pp. 61–68.
https://doi.org/10.2514/1.G001167

[5] Saleem, A., and Ratnoo, A., "Lyapunov-Based Guidance Law for Impact Time Control and Simultaneous Arrival," *Journal of Guidance, Control, and Dynamics*, Vol. 39, No. 1, 2016, pp. 164–173.
https://doi.org/10.2514/1.G001349

[6] Tekin, R., Erer, K. S., and Holzapfel, F., "Polynomial Shaping of the Look Angle for Impact-Time Control," *Journal of Guidance, Control, and Dynamics*, Vol. 40, No. 10, 2017, pp. 2668–2673.
https://doi.org/10.2514/1.G002751

[7] Tekin, R., Erer, K. S., and Holzapfel, F., "Adaptive Impact Time Control via Look-Angle Shaping Under Varying Velocity," *Journal of Guidance, Control, and Dynamics*, Vol. 40, No. 12, 2017, pp. 3247–3255.
https://doi.org/10.2514/1.G002981

[8] Tsalik, R., and Shima, T., "Circular Impact-Time Guidance," *Journal of Guidance, Control, and Dynamics*, Vol. 42, No. 8, 2019, pp. 1836–1847.
https://doi.org/10.2514/1.G004074

[9] Jeon, I.-S., Lee, J.-I., and Tahk, M.-J., "Homing Guidance Law for Cooperative Attack of Multiple Missiles," *Journal of Guidance, Control, and Dynamics*, Vol. 33, No. 1, 2010, pp. 275–280.
https://doi.org/10.2514/1.40136

[10] Zhou, J., and Yang, J., "Distributed Guidance Law Design for Cooperative Simultaneous Attacks with Multiple Missiles," *Journal of Guidance, Control, and Dynamics*, Vol. 39, No. 10, 2016, pp. 2439–2447.
https://doi.org/10.2514/1.G001609

[11] He, S., Wang, W., Lin, D., and Lei, H., "Consensus-Based Two-Stage Salvo Attack Guidance," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 54, No. 3, 2017, pp. 1555–1566.
https://doi.org/10.1109/TAES.2017.2773272

[12] Marshall, J. A., Broucke, M. E., and Francis, B. A., "Pursuit Formations of Unicycles," *Automatica*, Vol. 42, No. 1, 2006, pp. 3–12.
https://doi.org/10.1016/j.automatica.2005.08.001

[13] Dimarogonas, D. V., and Kyriakopoulos, K. J., "On the Rendezvous Problem for Multiple Nonholonomic Agents," *IEEE Transactions on Automatic Control*, Vol. 52, No. 5, 2007, pp. 916–922.
https://doi.org/10.1109/TAC.2007.895897

[14] Zheng, R., and Sun, D., "Rendezvous of Unicycles: A Bearings-Only and Perimeter Shortening Approach," *Systems & Control Letters*, Vol. 62, No. 5, 2013, pp. 401–407.
https://doi.org/10.1016/j.sysconle.2013.02.006

[15] Xiao, L., Boyd, S., and Kim, S.-J., "Distributed Average Consensus with Least-Mean-Square Deviation," *Journal of Parallel and Distributed Computing*, Vol. 67, No. 1, 2007, pp. 33–46.
https://doi.org/10.1016/j.jpdc.2006.08.010

[16] Nejad, B. M., Attia, S. A., and Raisch, J., "Max-Consensus in a Max-Plus Algebraic Setting: The Case of Fixed Communication Topologies," *2009 XXII International Symposium on Information, Communication and Automation Technologies*, Inst. of Electrical and Electronics Engineers, New York, 2009, pp. 1–7.
https://doi.org/10.1109/ICAT.2009.5348437

[17] Zhang, Y., and Li, S., "Distributed Biased Min-Consensus with Applications to Shortest Path Planning," *IEEE Transactions on Automatic Control*, Vol. 62, No. 10, 2017, pp. 5429–5436.
https://doi.org/10.1109/TAC.2017.2694547

[18] Bruckstein, A. M., "Why the Ant Trails Look so Straight and Nice," *Mathematical Intelligencer*, Vol. 15, No. 2, 1993, pp. 59–62.
https://doi.org/10.1007/BF03024195

[19] Kumar, S. R., and Mukherjee, D., "Cooperative Salvo Guidance Using Finite-Time Consensus over Directed Cycles," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 56, No. 2, 2019, pp. 1504–1514.
https://doi.org/10.1109/TAES.2019.2934675

[20] Jadbabaie, A., Lin, J., and Morse, A. S., "Coordination of Groups of Mobile Autonomous Agents Using Nearest Neighbor Rules," *Proceedings of the 41st IEEE Conference on Decision and Control, 2002*, Vol. 3, Inst. of Electrical and Electronics Engineers, New York, 2002, pp. 2953–2958.
https://doi.org/10.1109/TAC.2003.812781

[21] Sun, X., Zhou, R., Hou, D., and Wu, J., "Consensus of Leader-Followers System of Multi-Missile with Time-Delays and Switching Topologies," *Optik-International Journal for Light and Electron Optics*, Vol. 125, No. 3, 2014, pp. 1202–1208.
https://doi.org/10.1016/j.ijleo.2013.07.159

[22] Rudin, W., *Principles of Mathematical Analysis*, Vol. 3, McGraw–Hill, New York, 1964, pp. 89–98.

[23] Tekin, R., Erer, K. S., and Holzapfel, F., "Control of Impact Time with Increased Robustness via Feedback Linearization," *Journal of Guidance, Control, and Dynamics*, Vol. 39, No. 7, 2016, pp. 1682–1689.
https://doi.org/10.2514/1.G001719