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PURSUIT-EVASION GUIDANCE IN A SWITCHED SYSTEM*

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Abstract. A pursuit-evasion problem for an interceptor (pursuer) and a maneuverable target (evader) is considered. It is assumed that during the engagement the system overcomes multiple abrupt changes. This leads to a formulation of a pursuit-evasion game for a switched piecewise-linear system. In the case of complete information on the switch timing and on the system matrices, the differential game is solved based on the zero-effort miss distance in the switched system. In the case where switch moments and system matrices are unknown to one of the players, two matrix games (pursuit game and evasion game) are formulated.

Key words. interception problem, pursuit-evasion differential game, switched system, matrix game

AMS subject classifications. 49N70, 93C30, 91A10

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1. Introduction.

Recent decades show an increasing interest to switched (hybrid) systems in the control community. It is motivated by the fact that real-life systems can overcome abrupt changes of dynamics parameters, which can be caused either by deliberate control actions or by unpredictable external influence such as technical faults. Control problems in switched systems naturally arise in numerous applications, e.g., in supervisory industrial control [22, 17], robotics [18], automotive control [13], aerospace [29, 30, 34], and others.

There exists a wide literature on one-sided control problems in switched systems. Such problems were treated from various theoretic viewpoints both in stochastic and deterministic formulations (see, e.g., [8] and [21], respectively). The stability of switched systems were considered, e.g., by [1, 5, 20]. Necessary and sufficient optimality conditions were derived by [32, 4, 3, 31] based on specific maximum principle formulations and variational calculus considerations. There exists a wide literature on application of the viability theory to hybrid control systems (see, e.g., [15] and references therein). In particular, the viability approach was realized by using impulse differential inclusions in [2, 9]. Most papers on the optimal control in switched systems interpret a control strategy as a triplet consisting of (i) switch timing, (ii) dynamics sequence, and (iii) control function.

The number of papers on the switched control problems with more than a single participant (players or adversaries) is considerably smaller. A game theoretic approach to control problems in hybrid systems was developed by [33] based on hybrid automata dynamics. The application of viability theory to hybrid (impulsive) differential games is outlined in the survey [7]. Pursuit-evasion differential games with hybrid dynamics players, modeling a planar missile engagement with switched vehicle dynamics, were analyzed by [28, 29, 30, 11] for the case of a single dynamics

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switch. These efforts were motivated by the fact that modern intercepting vehicles can switch, e.g., aerodynamic control to thrust vector control, yielding hybrid system dynamics. The analysis of hybrid dynamics games was based on the known solutions of differential games with nonswitched dynamics and bounded controls [27, 26].

Dynamics switches can occur mainly for two reasons. First, it can be initiated deliberately by one or more adversaries (switching, for example, from a thrust vector control to an aerodynamic control). The second cause is an unpredictable technical fault of a control system, leading to the loss of the interceptor's advantage in maneuverability and/or agility [19]. Any malfunction in the aerodynamic steering system (breaking a control fin, a failure of one of the nozzles) can lead to an increase of the controller time constant, reducing the pursuer's agility. This is especially true for an aerodynamic control with multiple control surfaces (see, e.g., [39] and references therein). A dual canard-tail missile control system is considered in [25].

In this paper, a planar linearized engagement of an interceptor (pursuer) and a target (evader) is modeled as a pursuit-evasion differential game. It is assumed that during the engagement, the system overcomes multiple dynamics switches. First, the differential game is solved in the case of complete information on the switch moments and dynamics sequence, available to both players. To this end, the zero-effort miss distance (ZEM) is calculated for a switched system, providing the basis for the players' control strategies. The case of incomplete information is treated as a worst case from the player's viewpoint: the choice of actual switch moments and dynamics sequence (from a finite set of possibilities) is delegated to its counterpart, whereas the player's strategy is based on assumed timing and dynamics sequence. This allows treating both the pursuit and the evasion problems as matrix games. Illustrative numerical examples are presented.

2. Problem statement and preliminaries.

2.1. Engagement model. Planar engagement between a pursuer and an evader is considered. It is assumed that both adversaries have constant speeds.

In Figure 1, the schematic engagement geometry is depicted. The X-axis is the initial line of sight. The Y-axis is normal to the X-axis. The origin of the coordinate system is collocated with the pursuer's initial position. The points (x_p, y_p) and (x_e, y_e) are current coordinates of the pursuer and the evader, respectively; a_p, a_e are their lateral accelerations; φ_p, φ_e are the respective angles between the velocity vectors and

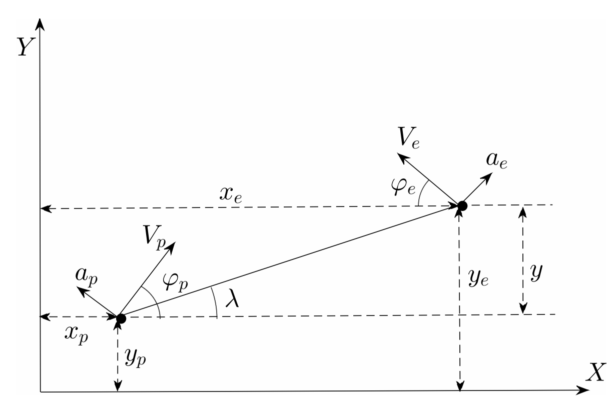

FIG. 1. *Engagement geometry.*

the X-axis; and λ is the line-of-sight angle. The aspect angles and the line-of-sight angle are assumed to be small during the engagement. This allows linearizing the relative trajectory with respect to the nominal collision geometry [38] and calculating the engagement duration

$$t_f = r(0)/(V_p + V_e), \tag{1}$$

where $r(t)$ is the distance between the target and the missile for $t \geq 0$. By (1), the zero separation in the X-direction between the missile and the target for $t = t_f$ is guaranteed. The separation in the Y-direction is $y = y_e - y_p$.

As in [24], it is assumed that for $t \in [0, t_f]$, the controller dynamics of the pursuer and the evader are described by linear differential equations

$$\dot{x}_k = A_k x_k + B_k u_k, \quad x_k(0) = x_{k0}, \quad k = p, e, \tag{2}$$

$$a_k = C_k x_k + d_k u_k, \quad k = p, e, \tag{3}$$

where x_k is the state vector consisting of n_k internal variables, u_k is the scalar control, a_k are the lateral accelerations (see Figure 1), and $k = p, e$. For example, an ideal player is given by $n_k = 0$, $A_k = B_k = C_k = 0$, $d_k = 1$, whereas for the player with the first-order strictly proper dynamics, $n_k = 1$, $A_k = -1/\tau_p$, $B_k = 1/\tau_p$, $C_k = 1$, $d_k = 0$, where τ_k is the controller time constant.

Let us define the state vector

$$x = [x_1, x_2, x_3, \ldots, x_{n_p+2}, x_{n_p+3}, \ldots, x_{n_p+n_e+2}]^T = [y \quad \dot{y} \quad x_p^T \quad x_e^T]^T \in \mathbb{R}^{n_p+n_e+2}, \tag{4}$$

where y is the relative separation (see Figure 1). Then, due to (2)–(3) and the small angles assumption [27], the system dynamics for $t \in [0, t_f]$ is described by the linear differential equation

$$\dot{x} = Ax + Bu_p + Cu_e, \tag{5}$$

where

$$A \triangleq \begin{bmatrix} 0 & 1 & [0] & [0] \\ 0 & 0 & -C_p & C_e \\ [0] & [0] & A_p & [0] \\ [0] & [0] & [0] & A_e \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0 \\ -d_p \\ B_p \\ [0] \end{bmatrix}, \quad C \triangleq \begin{bmatrix} 0 \\ d_e \\ [0] \\ B_e \end{bmatrix}; \tag{6}$$

[0] denotes a zero matrix of appropriate dimension. The initial condition is

$$x(0) = [0, x_{20}, x_{p0}^T, x_{e0}^T]. \tag{7}$$

Due to the small angles assumption (see section 2.1), $\sin\varphi_p \approx \varphi_p$, $\sin\varphi_e \approx \varphi_e$, and the initial relative velocity is $x_{20} = V_e\varphi_e(0) - V_p\varphi_p(0)$. The players' controls satisfy the constraints

$$|u_p| \leq u_p^{\max}, \quad |u_e| \leq u_e^{\max}. \tag{8}$$

Subject to these constraints, the pursuer minimizes, while the evader maximizes the miss distance

$$J = |y(t_f)| = |x_1(t_f)|. \tag{9}$$

2.2. Problem scalarization for constant dynamics. The problem (5), (9) can be scalarized by applying the well-known terminal projection transformation [6, 14] of the state variable: $Z(t,x) = D\Phi(t_f,t)x$, where

$$D = [1, 0, [0]_{1\times n_p}, [0]_{1\times n_e}]; \tag{10}$$

$\Phi(t,\tau)$ is the transition matrix of a homogeneous system $\dot{x} = Ax$. The variable $z(t) = Z(t,x(t))$ is called the ZEM, because it is equal to the miss distance created if the players' controls are zero from t to t_f. The ZEM has two remarkable features. First, it satisfies an autonomous differential equation, obtained by a direct differentiation and using the fact that a transition matrix $\Phi(t_f,t)$ satisfies $\dot{\Phi} = -\Phi A$: $\dot{z} = D(\dot{\Phi}x + \Phi\dot{x}) = D(-\Phi Ax + \Phi Ax + \Phi Bu_p + \Phi Cu_e) = h_1(t)u_p + h_2(t)u_e$, where $h_1(t) \triangleq D\Phi(t_f,t)B$, and $h_2(t) \triangleq D\Phi(t_f,t)C$. Second, by using $\Phi(t_f,t_f) = I_{n_p+n_e+2}$, $z(t_f) = x_1(t_f)$, which allows rewriting the cost functional as $J = |z(t_f)|$. Thus, if $u_p^*(t,z)$ and $u_e^*(t,z)$ are the optimal strategies in the pursuit-evasion differential game for the ZEM, then the respective strategies $u_p^0(t,x) = u_p^*(t,Z(t,x))$ and $u_e^0(t,x) = u_e^*(t,Z(t,x))$ are optimal in the differential game (5), (9), (8).

2.3. Switched pursuit-evasion game. Let $0 < t_{\mathrm{sw}_1} < t_{\mathrm{sw}_2} < \cdots < t_{\mathrm{sw}_m} < t_f$ be the switch moments given a priory. They divide the interval $[0,t_f]$ into $m+1$ intervals: $I_1 = [0,t_{\mathrm{sw}_1})$, $I_i = [t_{\mathrm{sw}_i}, t_{\mathrm{sw}_{i+1}})$, $i = 2,\dots,m$, and $I_{m+1} = [t_{\mathrm{sw}_m}, t_f)$. Let us assume that the dynamics coefficients of the players, defined by matrices A_k, B_k, and C_k and numbers d_k, $k = p,e$, can undergo abrupt changes for $t = t_{\mathrm{sw}_i}$, $i = 1,\dots,m$, leading to the changes of the dynamics triplet $\mathcal{D} = (A,B,C)$ in (5):

$$\mathcal{D} = \mathcal{D}_i = (A_i, B_i, C_i), \quad t \in I_i. \tag{11}$$

Thus, the system (5) becomes a *switched* system

$$\dot{x} = A_i x + B_i u_p + C_i u_e, \quad t \in I_i, \quad i = 1,\dots,m+1. \tag{12}$$

In the switched pursuit-evasion differential game (SPEG), the pursuer minimizes, whereas the evader maximizes (9), subject to the system dynamics (12) and the constraints (8). In this paper, the SPEG is treated in different information patterns.

Remark 1. In general, the dimensions n_p and/or n_e also can change. This leads to a more complicated switched state-varying system, augmented by concatenation operators [36, 37]. Such systems are out of the scope of this paper.

2.4. Problem scalarization for switched dynamics. Similarly to the case of constant dynamics, the SPEG is scalarized by introducing the ZEM as a new state variable. Let $u_p(\tau) = u_e(\tau) = 0$ for $\tau \in [t,t_f)$, $t \in I_i$, $i = 1,\dots,m+1$. Then $x(t_{\mathrm{sw}_i}) = \Phi_i(t_{\mathrm{sw}_i},\bar{t})x(t)$, $x(t_{\mathrm{sw}_{i+1}}) = \Phi_{i+1}(t_{\mathrm{sw}_{i+1}}, t_{\mathrm{sw}_i})x(t_{\mathrm{sw}_i})$, $\dots$, and $x(t_f) = \Phi_{m+1}(t_f,t_{\mathrm{sw}_m})x(t_{\mathrm{sw}_m})$, where $\Phi_i(t,\tau)$ are the transition matrices of the homogeneous systems $\dot{x} = A_i x$, $i = 1,\dots,m+1$.

This yields the state variable transformation

$$Z(t,x) = \begin{cases} D\Phi_{m+1}(t_f,t_{\mathrm{sw}_m}) \prod_{j=i}^{m-1} \Phi_{m+i-j}(t_{\mathrm{sw}_{m+i-j}}, t_{\mathrm{sw}_{m+i-j-1}}) \\ \times \Phi_i(t_{\mathrm{sw}_i},t)x, \qquad t \in I_i, i = 1,\dots,m, \\ \\ D\Phi_{m+1}(t_f,t)x, \qquad t \in I_{m+1}. \end{cases} \tag{13}$$

As in a nonswitched case, the ZEM $z(t) = Z(t, x(t))$ satisfies the autonomous differential equation

$$\dot{z} = h_1(t)u_p + h_2(t)u_e, \quad z(0) = z_0, \tag{14}$$

where

$$h_1(t) = \begin{cases} D\Phi_{m+1}(t_f, t_{\mathrm{sw}_m}) \prod_{j=i}^{m-1} \Phi_{m+i-j}(t_{\mathrm{sw}_{m+i-j}}, t_{\mathrm{sw}_{m+i-j-1}}) \\ \times\, \Phi_i(t_{\mathrm{sw}_i}, t)B_i(t), & t \in I_i, i = 1, \dots, m, \\ D\Phi_{m+1}(t_f, t)B_{m+1}(t), & t \in I_{m+1}, \end{cases} \tag{15}$$

$$h_2(t) = \begin{cases} D\Phi_{m+1}(t_f, t_{\mathrm{sw}_m}) \prod_{j=i}^{m-1} \Phi_{m+i-j}(t_{\mathrm{sw}_{m+i-j}}, t_{\mathrm{sw}_{m+i-j-1}}) \\ \times\, \Phi_i(t_{\mathrm{sw}_i}, t)C_i(t), & t \in I_i, i = 1, \dots, m, \\ D\Phi_{m+1}(t_f, t)C_{m+1}(t), & t \in I_{m+1}, \end{cases} \tag{16}$$

$$z_0 = D\Phi_{m+1}(t_f, t_{\mathrm{sw}_m}) \prod_{j=1}^{m-1} \Phi_{m+1-j}(t_{\mathrm{sw}_{m+1-j}}, t_{\mathrm{sw}_{m-j}})\Phi_1(t_{\mathrm{sw}_1}, 0)x_0. \tag{17}$$

Due to (13) and (10), $z(t_f) = x_1(t_f)$, which allows rewriting the cost functional (9) as

$$J = |z(t_f)|. \tag{18}$$

3. Solution in the case of complete information.

In this section, it is assumed that the dynamics triplets $\mathcal{D}_i = (A_i, B_i, C_i)$, $i = 1, \dots, m+1$, as well as the switch moments t_{sw_i}, $i = 1, \dots, m$, are known to the players in advance.

3.1. Solution.

As in a nonswitched case, if $u_p^*(t, z)$ and $u_e^*(t, z)$ are the optimal strategies in the differential game (14), (18), and (8), then the respective strategies $u_p^0(t, x) = u_p^*(t, Z(t, x))$ and $u_e^0(t, x) = u_e^*(t, Z(t, x))$ are optimal in the SPEG.

Due to [12], this differential game is solved based on the decomposition of the state space $\mathcal{S} = [0, t_f] \times \mathbb{R}$ in the (t, z)-plane. The regular region $\mathcal{R}_1 \subseteq \mathcal{S}$ is constituted by the trajectories, generated by the optimal strategies

$$u_p^* = \begin{cases} -u_p^{\max} \operatorname{sign}(h_1(t)) \operatorname{sign}(z(t)), & z(t) \neq 0, \\ 0, & z(t) = 0, \end{cases} \tag{19}$$

$$u_e^* = \begin{cases} u_e^{\max} \operatorname{sign}(h_2(t)) \operatorname{sign}(z(t)), & z(t) \neq 0, \\ u_e^{\max}, & z(t) = 0. \end{cases} \tag{20}$$

In the singular region $\mathcal{R}_0 = \mathcal{S} \backslash \mathcal{R}_1$, the optimal strategies are arbitrary, subject to the constraints (8). The bang-bang strategies (19)–(20) are optimal for any $(t, z) \in \mathcal{S}$. As shown by [10], the structure of the singular region can be rather complicated, depending on the number of zeros of a so-called determining function

$$R(t) \triangleq u_p^{\max}|h_1(t)| - u_e^{\max}|h_2(t)|. \tag{21}$$

Remark 2. The solution of the differential game (14), (18), and (8) is in line with the ideas of [28, 29, 30, 11], dealing with a single switch of the dynamics, and of [34], dealing with multiple switches in optimal control problem.

For $(t, z) \in \mathcal{R}_1$, the game value depends on the initial position:

$$J^0(t, z) = J^0(t_{\mathrm{sw}_1}, \dots, t_{\mathrm{sw}_m}, \mathcal{D}_1, \dots, \mathcal{D}_{m+1}) = |z| + \int_t^{t_f} R(\xi) d\xi. \tag{22}$$

For $(t, z) \in \mathcal{R}_0$, the game value is constant.

Remark 3. Due to [23, 10], if

$$Z(t) \triangleq \int_t^{t_f} R(\xi) \mathrm{d}\xi \geq 0, \quad t \in [0, t_f], \tag{23}$$

the game value in $\mathcal{R}_0$ is zero. Thus, the closure of $\mathcal{R}_0$,

$$\mathcal{C} = \mathcal{C}(t_{\mathrm{sw}_1}, \dots, t_{\mathrm{sw}_m}, \mathcal{D}_1, \dots, \mathcal{D}_{m+1}) = \left\{(t, z): \ t \in [0, t_f], \ |z| \leq Z(t)\right\}, \tag{24}$$

is the robust capture zone, i.e., the set of initial positions, from which the capture is guaranteed by the optimal pursuer's strategy against any admissible evader's control.

3.2. Example: First-order players' dynamics. Let us consider the example where the pursuer and the evader have first-order strictly proper dynamics with time constants τ_p and τ_e, respectively. Then, in the system (5), the dynamics triplet $\mathcal{D}$ is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1/\tau_p & 0 \\ 0 & 0 & 0 & -1/\tau_e \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 1/\tau_p \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\tau_e \end{bmatrix}, \tag{25}$$

where $x_1 = y$, $x_2 = \dot{y}$, $x_3 = a_p$, and $x_4 = a_e$. The initial condition is

$$x(0) = x_0 = [0, x_{20}, x_{30}, x_{40}]^T, \tag{26}$$

where x_{30} and x_{40} are the initial accelerations of the pursuer and the evader, respectively.

In this example, $m = 2$, i.e., the system undergoes two changes during the engagement at $t = t_{\mathrm{sw}_1}$ and $t = t_{\mathrm{sw}_2}$. The dynamics triplet (25) is completely defined by the values of τ_p and τ_e. Assume that

$$\tau_k = \tau_{k_i}, \quad t \in I_i, \ i = 1, 2, 3, \ k = p, e. \tag{27}$$

The transition matrices are

$$\Phi_i(t_2, t_1) = \Phi(t_2, t_1, \tau_{p_i}, \tau_{e_i}) = \begin{bmatrix} 0 & t_2 - t_1 & -\psi(t_2, t_1, \tau_{p_i}) & \psi(t_2, t_1, \tau_{e_i}) \\ 0 & 1 & -\varphi(t_2, t_1, \tau_{p_i}) & \varphi(t_2, t_1, \tau_{p_i}) \\ 0 & 0 & \chi(t_2, t_1, \tau_{p_i}) & 0 \\ 0 & 0 & 0 & \chi(t_2, t_1, \tau_{e_i}) \end{bmatrix}, \tag{28}$$

where $\chi(t_2, t_1, \tau) \triangleq \exp(-(t_2 - t_1)/\tau)$, $\varphi(t_2, t_1, \tau) \triangleq \tau\,(1 - \chi(t_2, t_1, \tau))$, $\psi(t_2, t_1, \tau) \triangleq \tau\,(-\varphi(t_2, t_1, \tau) + (t_2 - t_1))$.

Due to (15), (16), and (28),

$$h_1(t) = -h(t, t_{\mathrm{sw}_1}, t_{\mathrm{sw}_2}, \tau_{p_1}, \tau_{p_2}, \tau_{p_3}), \quad h_2(t) = h(t, t_{\mathrm{sw}_1}, t_{\mathrm{sw}_2}, \tau_{e_1}, \tau_{e_2}, \tau_{e_3}), \tag{29}$$

where

(30)
$$h(t,\theta_1,\theta_2,\tau_1,\tau_2,\tau_3) \triangleq \begin{cases} \frac{1}{\tau_1}\Big[f(\theta_1,t,\tau_1)+\chi(\theta_1,t,\tau_1) \\ \times\big(f(\theta_2,\theta_1,\tau_2)+\chi(\theta_2,\theta_1,\tau_2)\psi(t_f,\theta_2,\tau_3)\big)\Big], & t\in[0,\theta_1), \\ \frac{1}{\tau_2}\Big[f(\theta_2,t,\tau_2)+\chi(\theta_2,t,\tau_2)\psi(t_f,\theta_2,\tau_3)\Big], & t\in[\theta_1,\theta_2), \\ \frac{1}{\tau_3}\psi(t_f,t,\tau_3), & t\in[\theta_2,t_f), \end{cases}$$

$f(t_2,t_1,\tau) \triangleq \psi(t_2,t_1,\tau)+(t_2-t_1)\varphi(t_2,t_1,\tau)$.

Remark 4. In this example,

$$h_1(t)\le 0,\quad h_2(t)\ge 0,\quad t\in[0,t_f] \tag{31}$$

for any switch moments $t_{\mathrm{sw}_1} < t_{\mathrm{sw}_2}$ and for any time constants $\tau_{p_1},\tau_{p_2},\tau_{p_3}$ and $\tau_{e_1},\tau_{e_2},\tau_{e_3}$. Thus, the optimal strategies (19)–(20) reduce to

$$u_p^* = \begin{cases} u_p^{\max}\,\mathrm{sign}(z(t)), & z(t)\ne 0, \\ 0, & z(t)=0, \end{cases} \qquad u_e^* = \begin{cases} u_e^{\max}\,\mathrm{sign}(z(t)), & z(t)\ne 0, \\ u_e^{\max}, & z(t)=0. \end{cases} \tag{32}$$

Consider this example for $t_f = 4$ s, $t_{\mathrm{sw}_1} = 0.5$ s, $t_{\mathrm{sw}_2} = 1.5$ s, $\tau_{p_1} = 0.2$ s, $\tau_{p_2} = 0.1$ s, $\tau_{p_3} = 0.35$ s, $\tau_{e_1} = 0.3$ s, $\tau_{e_2} = 0.2$ s, $\tau_{e_3} = 0.1$ s, $u_p^{\max} = 120$ m/s^2, and $u_e^{\max} = 100$ m/s^2. For these parameters, the condition (23) is not valid (see Figure 2 depicting the function $Z(t)$, given in (23)).

In this example, the singular zone is

(33)
$$\mathcal{R}_0 = \mathcal{R}_0(t_{\mathrm{sw}_1},t_{\mathrm{sw}_2},\tau_{p_1},\tau_{e_1},\tau_{p_2},\tau_{e_2},\tau_{p_3},\tau_{e_3}) = \left\{(t,z) : t\in[0,t_s],\ |z|\le \int_t^{t_s} R(\xi)d\xi\right\},$$

where $t_s = 2.423$ s is the moment when $R(t_s) = 0$. The game space decomposition is shown in Figure 3. The interval $[t_s, t_f]$ is the dispersal line [16], meaning that

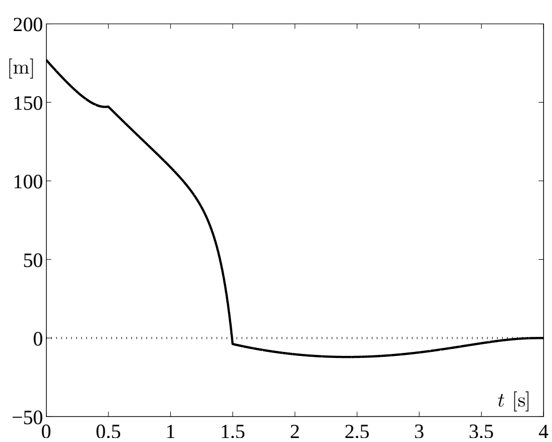

FIG. 2. *Function* $Z(t) : t_f = 4$ *s*, $t_{\mathrm{sw}_1} = 0.5$ *s*, $t_{\mathrm{sw}_2} = 1.5$ *s*, $\tau_{p_1} = 0.2$ *s*, $\tau_{e_1} = 0.3$ *s*, $\tau_{p_2} = 0.1$ *s*, $\tau_{e_2} = 0.2$ *s*, $\tau_{p_3} = 0.35$ *s*, $\tau_{e_3} = 0.1$ *s*, $u_p^{\max} = 120$ *m/s*^{2}, *and* $u_e^{\max} = 100$ *m/s*^{2}.

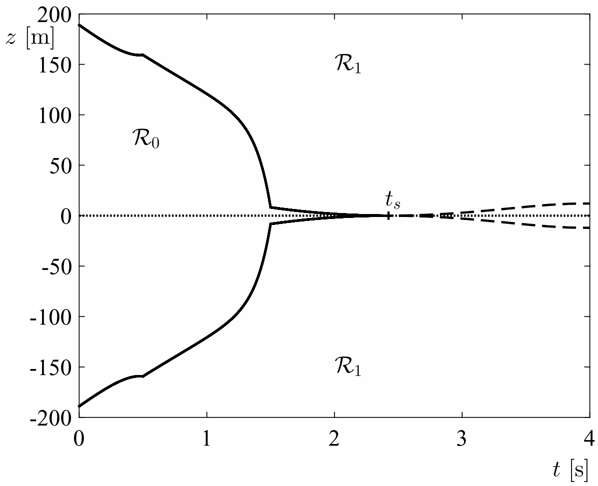

FIG. 3. *Game space decomposition:* $t_f = 4$ *s,* $t_{\mathrm{sw}_1} = 0.5$ *s,* $t_{\mathrm{sw}_2} = 1.5$ *s,* $\tau_{p_1} = 0.2$ *s,* $\tau_{e_1} = 0.3$ *s,* $\tau_{p_2} = 0.2$ *s,* $\tau_{e_2} = 0.2$ *s,* $\tau_{p_3} = 0.35$ *s,* $\tau_{e_3} = 0.1$ *s,* $u_p^{\max} = 120$ m/s^2*,* $u_e^{\max} = 100$ m/s^2.

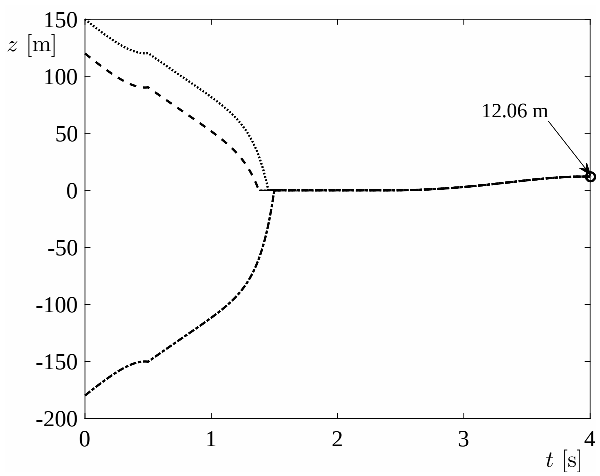

FIG. 4. *Optimal z-trajectories:* $t_f = 4$ *s,* $t_{\mathrm{sw}_1} = 0.5$ *s,* $t_{\mathrm{sw}_2} = 1.5$ *s,* $\tau_{p_1} = 0.2$ *s,* $\tau_{e_1} = 0.3$ *s,* $\tau_{p_2} = 0.1$ *s,* $\tau_{e_2} = 0.2$ *s,* $\tau_{p_3} = 0.35$ *s,* $\tau_{e_3} = 0.1$ *s,* $u_p^{\max} = 120$ m/s^2*,* $u_e^{\max} = 100$ m/s^2.

from the points $(t_0, 0)$, $t_0 \in [t_s, t_f]$, two symmetric optimal trajectories are generated, depending on the evader's choice $u_e = u_e^{\max}$ or $u_e = -u_e^{\max}$. At these trajectories the cost functional admits the same value. Two such trajectories, generated from $(t_s, 0)$, are shown in the dashed line.

In the singular zone $\mathcal{R}_0$, the game value is [10]

$$J^0(t, z) = \int_{t_s}^{t_f} |R(\xi)| d\xi \approx 12.06. \tag{34}$$

In Figure 4, three optimal z-trajectories of the system (14), defined by (29), are shown for three initial values $z_0 = 130$ m, $z_0 = 150$ m, and $z_0 = -180$ m. In all cases, $(0, z_0) \in \mathcal{R}_0$ and the miss distance is equal to 12.06 m, coinciding with the game value.

4. Unknown switch moments and dynamics order: Matrix game approach. In this section, the pursuit-evasion problem is treated for the case of incomplete information, separately from the viewpoints of the pursuer and the evader. It is assumed that the number of switch moments q and the set of possible dynamic triplets $\mathcal{D}_i = (A_i, B_i, C_i)$, $i = 1, \dots, M$, are known to both players in advance, whereas the actual switch moments $t_{\mathrm{sw}_1}, \dots, t_{\mathrm{sw}_q}$ and the actual dynamics sequence $\mathcal{D}_{i_1}, \dots, \mathcal{D}i_{q+1}$, $i_l \in \{1, \dots, M\}$, $l = 1, \dots, q+1$, are not. In contrast with this, let us assume that this information is available for the player's adversary, which constitutes the worst case from the player's viewpoint. This allows interpreting the pursuit and the evasion problems as two separate matrix games. In the first, pursuit, game, the role of the nature is delegated to the evader, whereas in the second, evasion, game, to the pursuer.

4.1. Players' strategies. Let us define the *strategies* of the pursuer and the evader as the pair of q-vector of switch moments and $(q+1)$-vector of dynamics indexes

$$U_k = \left\{ \left(t^k_{\mathrm{sw}_1}, \dots, t^k_{\mathrm{sw}_q}\right), \left(i^k_1, \dots, i^k_{q+1}\right) \right\}, \quad k = p, e, \tag{35}$$

where $i^k_l \in \{1, 2, \dots, M\}$, $t^k_{\mathrm{sw}_1} < t^k_{\mathrm{sw}_2} < \cdots < t^k_{\mathrm{sw}_q}$, $l = 1, \dots, q$, $k = p, e$. The indexes $i^k_1, \dots, i^k_{q+1}$, $k = p, e$, define the sequences of the dynamic triplets $\mathcal{D}_{i^k_1}, \dots, \mathcal{D}_{i^k_{q+1}}$. The switch moments $t^k_{\mathrm{sw}_1}, \dots, t^k_{\mathrm{sw}_q}$, $k = p, e$, generate the intervals I^k_i, $i = 1, \dots, q+1$.

The choice of a strategy U_k, $k = p, e$, means that the players employ their strategies (19)–(20), optimal in the game with complete information for the chosen sequences of switch moments and dynamic modes (*assumed as actual*):

$$u^*_p = \begin{cases} -u^{\max}_p \operatorname{sign}(h_{1p}(t)) \operatorname{sign}(z_p(t)), & z_p(t) \neq 0, \\ 0, & z_p(t) = 0, \end{cases} \tag{36}$$

$$u^*_e = \begin{cases} u^{\max}_e \operatorname{sign}(h_{2e}(t)) \operatorname{sign}(z_e(t)), & z_e(t) \neq 0, \\ u^{\max}_e, & z_e(t) = 0, \end{cases} \tag{37}$$

where for $k = p, e$,

$$z_k(t) = \begin{cases} D\Phi_{i^k_{q+1}}(t_f, t_{\mathrm{sw}^k_q}) \prod_{j=l}^{q-1} \Phi_{i^k_{q+l-j}}(t^k_{\mathrm{sw}_{q+l-j}}, t^k_{\mathrm{sw}_{q+l-j-1}}) & \\ \times \Phi_{i^k_l}(t_{\mathrm{sw}^k_i}, t) x(t), & t \in I^k_l, l = 1, \dots, q, \\ D\Phi_{i^k_{q+1}}(t_f, t) x(t), & t \in I^k_{q+1}, \end{cases} \tag{38}$$

$$h_{1k}(t) = \begin{cases} D\Phi_{i^k_{q+1}}(t_f, t_{\mathrm{sw}^k_q}) \prod_{j=l}^{q-1} \Phi_{i^k_{q+l-j}}(t^k_{\mathrm{sw}_{q+l-j}}, t^k_{\mathrm{sw}_{q+l-j-1}}) & \\ \times \Phi_{i^k_l}(t_{\mathrm{sw}^k_i}, t) B_{i^k_l}(t), & t \in I^k_l, l = 1, \dots, q, \\ D\Phi_{i^k_{q+1}}(t_f, t) B_{i^k_{q+1}}(t), & t \in I^k_{q+1}, \end{cases} \tag{39}$$

$$h_{2k}(t) = \begin{cases} D\Phi_{i^k_{q+1}}(t_f, t_{\mathrm{sw}^k_q}) \prod_{j=l}^{q-1} \Phi_{i^k_{q+l-j}}(t^k_{\mathrm{sw}_{q+l-j}}, t^k_{\mathrm{sw}_{q+l-j-1}}) & \\ \times \Phi_{i^k_l}(t_{\mathrm{sw}^k_i}, t) C_{i^k_l}(t), & t \in I^k_l, l = 1, \dots, q, \\ D\Phi_{i^k_{q+1}}(t_f, t) C_{i^k_{q+1}}(t), & t \in I^k_{q+1}. \end{cases} \tag{40}$$

Both players can choose their strategies from the sets

$$\mathcal{U}_k = \{U_{k_i}, i = 1, \dots, N_k\}, \quad k = p, e, \tag{41}$$

where for $k = p, e$, $i = 1, \dots, N_k$,

$$U_{k_i} = \left\{ \left(t^k_{\mathrm{sw}_{i,1}}, \dots, t^k_{\mathrm{sw}_{i,q}} \right), \left(i^k_{i,1}, \dots, i^k_{i,q+1} \right) \right\}. \tag{42}$$

4.2. Worst-case matrix games. In this section, two separate zero-sum matrix games [35, Chap. 3] are formulated. In each game, representing the worst case for one of the players (the pursuer or the evader), the choice of *actual* switch moments and the *actual* dynamics sequence is delegated to its adversary (to the evader or to the pursuer, respectively).

4.2.1. Pursuit game (Game I). In the pursuit game (worst case from the pursuer's viewpoint), the actual switch moments and the actual dynamics sequence are defined by the evader's strategy:

$$t_{\mathrm{sw}_l} = t^e_{\mathrm{sw}_l}, \ l = 1, \dots, q; \ \mathcal{D}_l = \mathcal{D}_{i^e_l}, \ l = 1, \dots, q+1. \tag{43}$$

Due to (37), this means that the evader acts optimally in the differential game with complete information, and the actual ZEM is $z_e(t)$. The pursuer does not know the actual dynamics sequence and relies on its strategy U_p, determining its feedback control $u^*_p(t, z)$ by (36).

Remark 5. It should be emphasized that the pursuer's feedback control $u_p = u^*_p(t, z)$ is the optimal strategy in the differential game with complete information (section 3.1), where the switch moments and the dynamics sequence are assumed by the pursuer. A differential game formulation where the pursuer chooses only a feedback strategy, in order to minimize the miss distance (18), whereas the evader chooses both the switching policy and the feedback strategy, in order to maximize (18), is not considered in this paper.

The pair of strategies (U_{e_i}, U_{p_j}), $i = 1, \dots, N_e$, $j = 1, \dots, N_p$, completely determines both the system switched dynamics and the players' controls. This generates the corresponding value of the cost functional (18): $J = J^I_{ij} = J(U_{e_i}, U_{p_j})$, yielding the $(N_e \times N_p)$-matrix

$$\mathcal{J}^I = \begin{array}{c} \begin{array}{cccc} U_{p_1} & U_{p_2} & \dots & U_{p_{N_p}} \end{array} \\ \begin{bmatrix} J^I_{11} & J^I_{12} & \dots & J^I_{1,N_p} \\ J^I_{21} & J^I_{22} & \dots & J^I_{2,N_p} \\ & & \vdots & \\ J^I_{N_e,1} & J^I_{N_e,2} & \dots & J^I_{N_e,N_p} \end{bmatrix} \end{array} \begin{array}{c} \\ U_{e_1} \\ U_{e_2} \\ \vdots \\ U_{e_{N_e}} \end{array} \tag{44}$$

Note that the matrix can be calculated off-line, before the engagement. Let us formulate the zero-sum matrix game [35, Chap. 3] over $\mathcal{J}^I$ with the pursuer as the minimizer and the evader as the maximizer, Game I. In this game, the lower value is

$$J^I_* = \max_{i=1,\dots,N_e} \min_{j=1,\dots,N_p} J^I_{ij}, \tag{45}$$

meaning the miss distance, guaranteed for the evader: no matter what strategy the pursuer uses, the miss distance will not be smaller than J^I_*. The evader's strategy

$U_{e_I}^{\max\min}$, for which the maximum in (45) is attained, is called the *maximin* strategy in Game I. Similarly, the upper value is

$$J_I^* = \min_{j=1,\dots,N_p} \max_{i=1,\dots,N_e} J_{ij}^I, \tag{46}$$

meaning the miss distance, guaranteed for the pursuer: no matter what strategy the evader uses, the miss distance will not be larger than J_I^*. The pursuer's strategy $U_{p_I}^{\min\max}$, for which the minimum in (46) is attained, is called the *minimax* strategy in Game I. Note that the maximin and the minimax strategies are not necessarily unique.

In general, $J_*^I \leq J_I^*$. If $J_*^I = J_I^* = J_I^0$, then the game has the saddle point, and the pair, consisting of the evader's maximin strategy $U_{e_I}^{\max\min} = U_{e_I}^0$ and the pursuer's minimax strategy $U_{p_I}^{\min\max} = U_{p_I}^0$, constitute the saddle point of the game. In this case, the saddle point inequality

$$J(U_e, U_{p_I}^0) \leq J(U_{e_I}^0, U_{p_I}^0) \leq J(U_{e_I}^0, U_p) \tag{47}$$

is valid for any admissible U_e, U_p.

4.2.2. Evasion game (Game II). The evasion game is formulated similar to the pursuit game by swapping the roles of the pursuer and the evader. In this game (worst case from the evader's viewpoint), the actual switch moments and the actual dynamics sequence are

$$t_{\mathrm{sw}_l} = t_{\mathrm{sw}_l}^p,\ l = 1,\dots,q; \mathcal{D}_l = \mathcal{D}_{i_l^p},\ l = 1,\dots,q+1, \tag{48}$$

meaning that the pursuer acts optimally in the game with complete information, and the actual ZEM is $z_p(t)$. The evader is not aware of the actual dynamics sequence and relies on its strategy U_e, determining its feedback control $u_e^*(t,z)$ by (37). The case is symmetric to that in Game I (see Remark 5): the evader's feedback strategy $u_e = u_e^*(t,z)$ is optimal in the differential game with complete information where the switch moments and the dynamics sequence are assumed by the evader. The game formulation where the evader chooses only a feedback strategy, whereas the pursuer chooses both the switching policy and the feedback strategy, is not considered in this paper.

Similar to Game I, the pair of strategies (U_{e_i}, U_{p_j}) generates the value of the cost functional (18): $J = J_{ij}^{II} = J(U_{e_i}, U_{p_j})$, $i = 1,\dots,N_e$, $j = 1,\dots,N_p$, and the corresponding game matrix $\mathcal{J}^{II}$. The lower and the upper values J_*^{II} and J_{II}^* in this game are defined as in (45) and (46) by replacing J_{ij}^I with J_{ij}^{II}. The evader's maximin strategy $U_{e_{II}}^{\max\max}$, the pursuer's minimax strategy $U_{p_{II}}^{\min\max}$, the game value J_0^{II}, and the saddle point $(U_{e_{II}}^0, (U_{p_{II}}^0)$ are defined similar to Game I.

4.3. Properties of Games I and II.

4.3.1. Basic properties. Let for $i = 1,\dots,N_e$,

$$J_{I_i}^0 = J^0(t_0, z_0, t_{\mathrm{sw}_{i,1}}^e, \dots, t_{\mathrm{sw}_{i,q}}^e, \mathcal{D}_{i_{i,1}^e}, \dots, \mathcal{D}_{i_{i,q+1}^e}) \tag{49}$$

be the values of the games with complete information, generated by the evader's strategies in Game I. Similarly, let for $j = 1,\dots,N_p$,

$$J_{II_j}^0 = J^0(t_0, z_0, t_{\mathrm{sw}_{j,1}}^p, \dots, t_{\mathrm{sw}_{j,q}}^p, \mathcal{D}_{i_{j,1}^p}, \dots, \mathcal{D}_{i_{j,q+1}^p}) \tag{50}$$

be the values of the games with complete information, generated by the pursuer's strategies in Game II.

PROPOSITION 1. *Let there exist $j = j_0$ such that $J^0_{II_{j_0}} = 0$ in the singular zone $\mathcal{R}_0$. If*

$$(t_0, z_0) \in \mathcal{C}(t^p_{\mathrm{sw}_{j_0,1}}, \ldots, t^p_{\mathrm{sw}_{j_0,q}}, \mathcal{D}_{i^p_{j_0,1}}, \ldots, \mathcal{D}_{i^p_{j_0,q+1}}), \tag{51}$$

then Game II *has the saddle point and its value is zero.*

Proof. Since in Game II the pursuer acts optimally in the game with complete information, the entries of the matrix $\mathcal{J}^{II}$ satisfy the inequality

$$J^{II}_{ij} \leq J^0_{II_j},\ i = 1, \ldots, N_e, \tag{52}$$

for all $j = 1, \ldots, N_p$, $(t_0, z_0) \in \mathcal{S}$. In particular, for $j = j_0$, $(t_0, z_0) \in \mathcal{C}$:

$$J^{II}_{ij_0} \leq J^0_{II_{j_0}} = 0,\ i = 1, \ldots, N_e. \tag{53}$$

Thus,

$$J^{II}_{ij_0} = 0,\ i = 1, \ldots, N_e, \tag{54}$$

i.e., the matrix $\mathcal{J}^{II}$ has the zero column. This means that

$$J^{II}_* = J^*_{II} = 0. \tag{55}$$ ∎

PROPOSITION 2. *If for some initial position $(t_0, z_0) \in \mathcal{S}$,*

$$\max_{i=1,\ldots,N_e} J^0_{I_i} \geq \min_{j=1,\ldots,N_p} J^0_{II_j}, \tag{56}$$

then for this position,

$$J^I_* \geq J^*_{II}. \tag{57}$$

Proof. Since in Game I the evader acts optimally in the game with complete information, the entries of the matrix $\mathcal{J}^I$ satisfy the inequality

$$J^I_{ij} \geq J^0_{I_i},\ j = 1, \ldots, N_p, \tag{58}$$

for all $i = 1, \ldots, N_e$, $(t_0, z_0) \in \mathcal{S}$. Therefore,

$$J^I_* = \max_{i=1,\ldots,N_e} \min_{j=1,\ldots,N_p} J^I_{ij} \geq \max_{i=1,\ldots,N_e} J^0_{I_i}. \tag{59}$$

Due to (52),

$$J^*_{II} = \min_{j=1,\ldots,N_p} \max_{i=1,\ldots,N_e} J^{II}_{ij} \leq \min_{j=1,\ldots,N_p} J^0_{II_j}. \tag{60}$$

The inequalities (59), (60), and (56) directly yield (57). ∎

Due to Proposition 2 and definitions of lower and upper game values, $J^*_I \geq J^I_* \geq J^*_{II} \geq J^{II}_*$.

Remark 6. The condition (56) is valid for any $(t_0, z_0) \in \mathcal{S}$ if the sets of strategies of the pursuer and the evader coincide:

$$\mathcal{U}_p = \mathcal{U}_e = \mathcal{U} = \{U_1, \ldots, U_N\}. \tag{61}$$

In this case, inequalities in (59) and (60) become equalities:

$$J^I_* = \max_{i=1,\ldots,N_e} J^0_{I_i}, \quad J^*_{II} = \min_{j=1,\ldots,N_p} J^0_{II_j}. \tag{62}$$

4.3.2. Simplest case. Let us consider the simplest case of the matrix games I and II, where the pursuer and the evader choose their strategies from the same set of two strategies ($N_p = N_e = 2$):

$$\mathcal{U}_p = \mathcal{U}_e = \{U_1, U_2\}, \tag{63}$$

where for $i = 1, 2$,

$$U_i = \left\{ \left(t_{\mathrm{sw}_{i,1}}, \ldots, t_{\mathrm{sw}_{i,q}}\right), (i_{i,1}, \ldots, i_{i,q+1}) \right\}. \tag{64}$$

In this case, the matrices $\mathcal{J}^I$ and $\mathcal{J}^{II}$ are 2×2. The diagonal elements of these matrices correspond to the case where both players act optimally in the game with complete information, yielding

$$\begin{aligned} J_{ii}^I &= J_{ii}^{II} = J_i^0 \\ &= J^0\left(t_0, z_0, t_{\mathrm{sw}_{i,1}}, \ldots, t_{\mathrm{sw}_{i,q}}, \mathcal{D}_{i_{i,1}}, \ldots, \mathcal{D}_{i_{i,q+1}}\right), \ i = 1, 2. \end{aligned} \tag{65}$$

Thus, the game matrices have a form

$$\mathcal{J}^k = \begin{bmatrix} J_1^0 & J_{12}^k \\ J_{21}^k & J_2^0 \end{bmatrix}, \quad k = I, II. \tag{66}$$

Without loss of generality, we assume that

$$J_1^0 \le J_2^0. \tag{67}$$

PROPOSITION 3. *In Game I,*

$$J_*^I = J_2^0, \tag{68}$$

$$J_I^* = \begin{cases} \min\{J_{12}^I, J_{21}^I\}, & \text{if } J_{12}^I > J_2^0, \\ J_2^0, & \text{if } J_{12}^I \le J_2^0. \end{cases} \tag{69}$$

Proof. Since in Game I the actual dynamics sequence is determined by the evader's strategy, the values J_{12}^I and J_{21}^I are obtained when the evader acts optimally in the game with complete information, whereas the pursuer does not, i.e.,

$$J_{12}^I \ge J_1^0, \quad J_{21}^I \ge J_2^0. \tag{70}$$

Therefore, the minimum values over the rows are

$$\min_{j=1,2} J_{ij}^I = J_i^0, \ i = 1, 2, \tag{71}$$

implying, by virtue of the assumption (67), that

$$J_*^I = \max_{i=1,2} \min_{j=1,2} J_{ij} = \max\{J_1^0, J_2^0\} = J_2^0. \tag{72}$$

Thus, (68) is justified.

Due to (67) and (70),

$$J_{21}^I \ge J_1^0, \tag{73}$$

i.e., the maximum in the first column is

$$\max_{i=1,2} J_{i1}^{I} = J_{21}^{I}. \tag{74}$$

The maximum in the second column is

$$\max_{i=1,2} J_{i2}^{I} = \begin{cases} J_{12}^{I}, & \text{if } J_{12}^{I} > J_2^0, \\ J_2^0, & \text{if } J_{12}^{I} \le J_2^0. \end{cases} \tag{75}$$

Equations (74)–(75), along with (70), directly yield (69). ∎

Remark 7. If $J_{12}^{I} \le J_2^0$, then Game I has the saddle point, and its value is equal to J_2^0.

PROPOSITION 4. *In Game* II*,*

$$J_*^{II} = \begin{cases} \max\{J_{12}^{II}, J_{21}^{II}\}, & \text{if } J_{12}^{II} < J_1^0, \\ J_1^0, & \text{if } J_{12}^{II} \ge J_1^0, \end{cases} \tag{76}$$

$$J_{II}^{*} = J_1^0. \tag{77}$$

Proof. The proposition is proved similarly to Proposition 3, based on the inequalities

$$J_{12}^{II} \le J_2^0, \quad J_{21}^{II} \le J_1^0. \tag{78}$$ ∎

Remark 8. If $J_{12}^{II} \ge J_1^0$, then Game II has the saddle point, and its value is equal to J_1^0.

4.4. Numerical examples.

4.4.1. Examples with two strategies. Consider the example of section 3.2, where the system is described by (5) with the matrices (25). In this example, the dynamic triple $\mathcal{D}$ is completely determined by a pair (τ_p, τ_e) of the time constants of the pursuer and the evader. For two switches ($q = 2$), the time constants change in accordance with (27). A strategy notation can be naturally simplified as

$$U = \{(t_{\mathrm{sw}_1}, t_{\mathrm{sw}_2}), (\tau_{p_1}, \tau_{e_1}), (\tau_{p_2}, \tau_{e_2}), (\tau_{p_3}, \tau_{e_3})\}, \tag{79}$$

where, instead of the dynamics numbers i_{i1}, i_{i2}, and i_{i3}, the actual values of the time constants are indicated. For $t_f = 4$ s, $u_p^{\max} = 130$ m/s^2, $u_e^{\max} = 100$ m/s^2, $x_{20} = 20$ m/s, $x_{30} = x_{40} = 0$ m/s^2, and for the strategies

$$U_1 = \{(1.5, 3), (0.5, 0.25), (0.4, 0.15), (0.15, 0.3)\}, \tag{80}$$
$$U_2 = \{(1, 2.5), (0.25, 0.55), (0.35, 0.45), (0.5, 0.25)\}, \tag{81}$$

the game matrix in Game I is

$$\mathcal{J}^I = \begin{bmatrix} 0 & 12.75 \\ 8.71 & 3.13 \end{bmatrix}. \tag{82}$$

In this example, $J_1^0 = 0$ m, i.e., in the game with complete information, corresponding to the strategy U_1, there exists the robust capture zone which can be denoted as $\mathcal{C}(U_1)$. In Figure 5, the z_e-trajectories, corresponding to the first row of the matrix (82), are

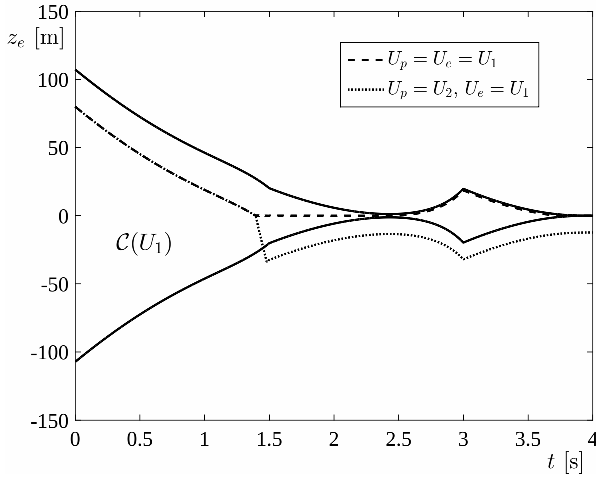

FIG. 5. *Trajectories z_e in Game* I *corresponding to the first row of the matrix* (82).

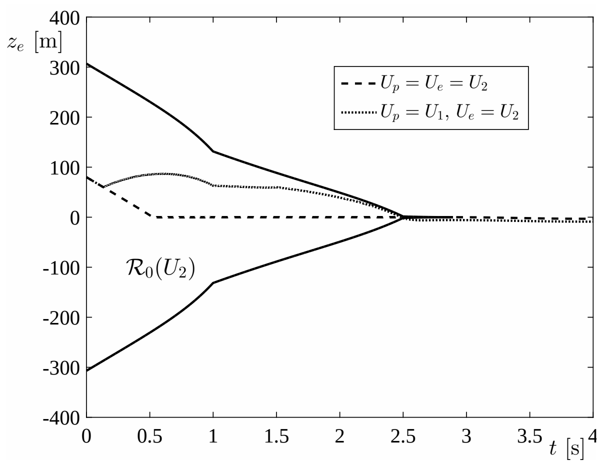

FIG. 6. *Trajectories z_e in Game* I *corresponding to the second row of the matrix* (82).

shown in dashed and dotted lines, as well as the boundaries of the robust capture zone shown in solid lines.

It is seen that if both adversaries stick to their optimal strategies in the game with complete information, corresponding to U_1, the miss distance is equal to the game value J_1^0 (the trajectory is shown in the dashed line). If the pursuer deviates from its optimal strategy (the trajectory shown in the dotted line), the game output is larger than J_1^0.

In the game with complete information, corresponding to the strategy U_2, the game value is $J_2^0 = 3.13$ m. In Figure 6, the z_e-trajectories, corresponding to the second row of the matrix (82), are shown, along with the boundaries of the singular zone $\mathcal{R}_0(U_2)$, in solid lines. The singular zone is given by (33) for $t_s = 2.89$ s.

The trajectory, shown in the dashed line, is generated by the optimal strategies in the game with complete information corresponding to U_2. It passes through the dispersion point $(t_s, 0)$ and then coincides with one of the optimal trajectories, emerging

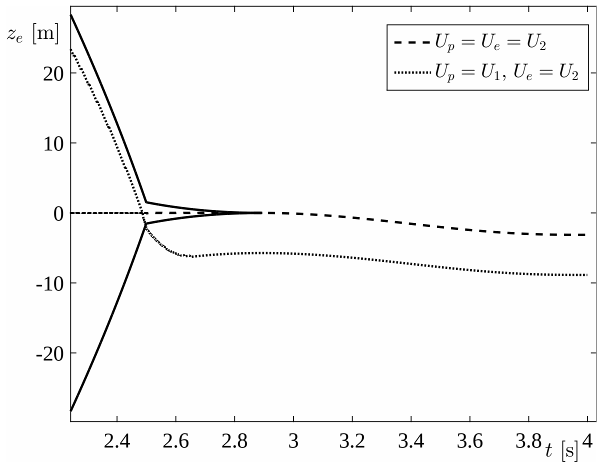

FIG. 7. *Trajectories z_e (zoomed) in Game* I *corresponding to the second row of the matrix* (82).

from this point (see the zoomed Figure 7). In this case, the miss distance is equal to J_2^0 m. If the pursuer deviates from its optimal strategy in the game with complete information (the trajectory shown in the dotted line) the miss distance is larger than J_2^0.

As is stated in (68), the lower value is equal to the largest value in the games with complete information: $J_*^I = J_2^0 = 3.12$ m. One can note that $J_{12}^I = 12.75 > J_2^0$, and, according to the first possibility in (69), $J_I^* = \min\{J_{12}^I, J_{21}^I\} = \min\{12.75, 8.71\} = 8.71$ m. In this example, Game I does not have the saddle point.

For the same values of t_f, $u_p^{\max}$, $u_e^{\max}$,x_{20}, x_{30}, x_{40} as in the previous example, and for the other pair of strategies

$$U_1 = \{(3, 3.5), (0.05, 0.3), (0.05, 0.5), (0.35, 0.25)\}, \tag{83}$$

$$U_2 = \{(2.5, 3), (0.4, 0.5), (0.5, 0.15), (0.35, 0.2)\}, \tag{84}$$

the game matrix in Game I is

$$\mathcal{J}^I = \begin{bmatrix} 0.01 & 0.57 \\ 36.44 & 10.15 \end{bmatrix}. \tag{85}$$

In this example, $J_{12}^I = 0.57 < J_2^0 = 10.15$, and the game has the saddle point $J_I^0 = J_*^I = J_I^* = 10.15$ m.

For the same values of t_f, $u_p^{\max}$, $u_e^{\max}$, x_{30}, x_{40} as in the previous examples, for $x_{20} = 10$ m/s, and for the strategies

$$U_1 = \{(1.5, 3.5), (0.5, 0.05), (0.25, 0.15), (0.25, 0.25)\}, \tag{86}$$

$$U_2 = \{(3, 3.5), (0.45, 0.1), (0.4, 0.45), (0.05, 0.5)\}, \tag{87}$$

the game matrix in Game II is

$$\mathcal{J}^{II} = \begin{bmatrix} 4.92 & 13.45 \\ 0 & 13.45 \end{bmatrix}. \tag{88}$$

In this example, in the game with complete information, corresponding to the strategy U_1, there exists the robust capture zone $\mathcal{C}(U_1)$. For $x_{20} = 10$ m/s, $z_0 = 40$ m, and

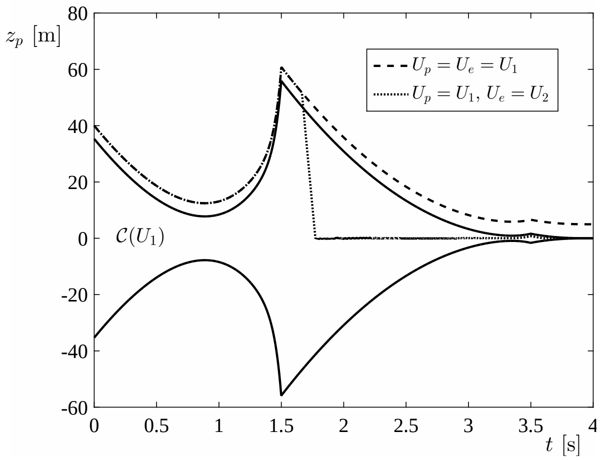

FIG. 8. *Trajectories z_p in Game* II *corresponding to the first column of the matrix* (88).

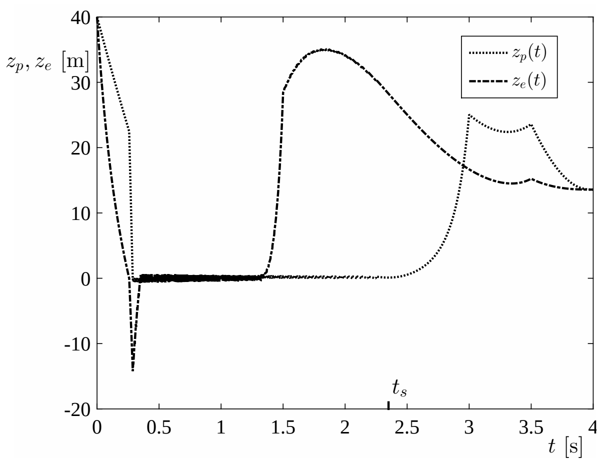

FIG. 9. *Trajectories $z_p(t)$ and $z_e(t)$ in Game* II *for $U_p = U_2$, $U_e = U_1$.*

$z_0 \notin \mathcal{C}(U_1)$, yielding $J_1^0 > 0$. In Figure 8, the z_p-trajectories, corresponding to the first column of the matrix (88), are shown, along with the boundaries of the robust capture zone. It is seen that for $U_p = U_e = U_1$ (optimal behavior of both adversaries in the game with complete information), the trajectory remains outside $\mathcal{C}(U_1)$ and terminates at the point $(4, 4.92)$. For $U_p = U_1$, $U_e = U_2$ (the evader does not act optimally in the game with complete information), the trajectory enters into the robust capture zone providing zero miss distance.

In Figure 8, the z_p-trajectories, corresponding to the first column of the matrix (88), are shown, along with the boundaries of the singular zone $\mathcal{R}_0(U_2)$. The singular zone is given by (33) for $t_s = 2.35$ s. Both for $U_p = U_e = U_2$ and for $U_p = U_2$, $U_e = U_1$, the trajectories pass through the dispersal point $(t_s, 0)$ and then coincide with each other.

It is explained by Figure 9 where the actual ZEM $z_p(t)$ and the function $z_e(t)$ used as ZEM by the evader are depicted. It is seen that these functions are positive for $t \geq t_s$. Therefore, the evader's control $u_e(t) = \text{sign}(z_e(t))$ after the dispersal

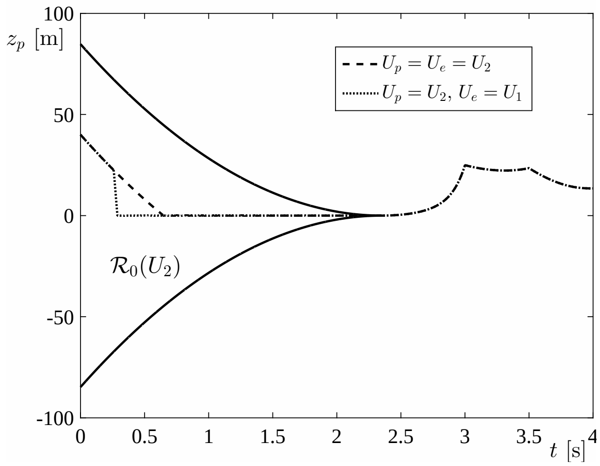

FIG. 10. *Trajectories* z_p *in Game* II *corresponding to the second column of the matrix* (88).

point coincides with the optimal control $u_e^*(t) = \text{sign}(z_p(t)$ in the game with complete information corresponding to U_2, and, respectively, the trajectory coincides with the optimal one as shown in Figure 10.

In this example, $J_{12}^{II} = 13.46 > J_1^0 = 4.92$, and the game has the saddle point $J_{II}^0 = J_*^{II} = J_{II}^* = 4.92$ m.

Remark 9. By a wide numeric search, in which the strategies U_1, U_2, the initial value x_{20}, and the maneuverability ratio $u_p^{\max}/u_e^{\max}$ were chosen randomly, no example was found, where $J_{12}^{II} < J_1^0$. Moreover, an example where both inequalities $J_{12}^{II} < J_2^0$ and $J_{21}^{II} < J_1^0$ are valid, also was not found.

4.4.2. Examples with numerous strategies. In this section, we continue considering the example where the system is described by (5) with the matrices (25). The system undergoes two dynamics switches. The switch moments $t_{\text{sw}_1}, t_{\text{sw}_2} \in (0, t_f)$, $t_{\text{sw}_1} < t_{\text{sw}_2}$, are all possible moments from the set $T = \{t_1, \ldots, t_{n_{\text{sw}}}\}$. The time constants τ_{p_i}, τ_{e_i}, $i = 1, 2, 3$, admit all possible values from the set $\mathcal{T} = \{\tau_1, \ldots, \tau_{n_\tau}\}$ satisfying the condition

$$(\tau_{p_i}, \tau_{e_i}) \neq (\tau_{p_{i+1}}, \tau_{e_{i+1}}), \quad i = 1, 2, \tag{89}$$

guaranteeing that at the switch moments the system dynamics actually changes. Assume that the adversaries choose their strategies from the same set $\mathcal{U}_p = \mathcal{U}_e = \mathcal{U} = \{U_1, \ldots, U_N\}$ where, due to (89),

$$N = \binom{n_{\text{sw}}}{2} n_\tau^2 (n_\tau^2 - 1)^2. \tag{90}$$

In Table 1, the values of Games I and II are presented for $t_f = 4$ s, $u_p^{\max} = 130$ m/s^2, $u_e^{\max} = 100$ m/s^2, $T = \{1, 2, 3\}$ ($n_{\text{sw}} = 3$), $\mathcal{T} = \{0.1, 0.3\}$ ($n_\tau = 2$), $x_{30} = 0$ m/s^2, $x_{40} = 0$ m/s^2, and for two values of x_{20}. In this example, number of strategies is $N = \binom{3}{2} \cdot 9 \cdot 4 = 108$.

It is seen that for $x_{20} = 30$ m/s, Game I does not have saddle point, whereas for $x_{20} = 35$ m/s, this game has the saddle point. In both cases, Game II has saddle point which is equal to 0. For $x_{20} = 30$ m/s, the minimax pursuer's strategy in Game I is

$$U_{p_I}^{\min\max} = \{1, 2, (0.1, 0.1), (0.3, 0.3), (0.3, 0.1)\}, \tag{91}$$

TABLE 1
Game values.

x_{20} [m/s]	J_*^I [m]	J_I^* [m]	J_*^{II} [m]	J_{II}^* [m]
30	4.11	3.85	0	0
35	5.35	5.35	0	0

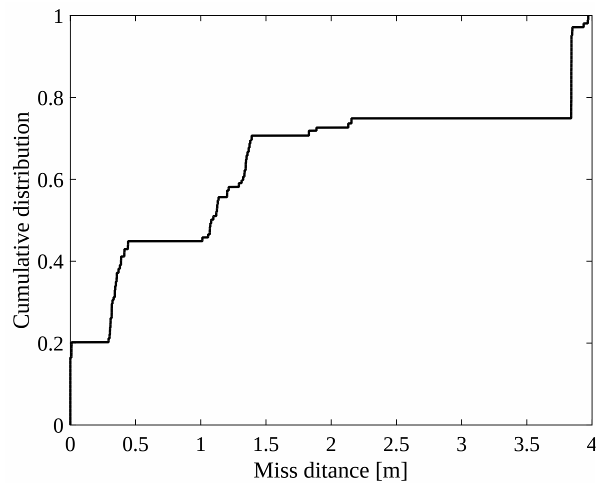

FIG. 11. *Cumulative distribution of miss distance from* 5000 *Monte Carlo simulations for* $U_p = U_{p_I}^{\min\max}$, $U_e = U_{e_{II}}^{\max\min}$, *and random* U_a.

and the maximin evader's strategy in Game II is

$$U_{e_{II}}^{\max\min} = \{1, 2, (0.1, 0.1), (0.1, 0.3), (0.3, 0.1)\}. \tag{92}$$

Let us consider the case where the actual switch moments and dynamics sequence (the "strategy of the nature" U_a) are independent of both adversaries, whereas the pursuer's and the evader's controls are based on some strategies U_p and U_e.

In Figure 11, the cumulative miss distance distribution is presented for the case where the pursuer employs its minimax strategy in Game I, the evader uses its maximin strategy in Game II, whereas "the strategy of the nature" U_a is chosen randomly from the set $\mathcal{U}$. It is seen that the miss distances fall into the interval $[J_*^{II}, J_I^*]$. In this simulation, the average miss distance is 1.46 m which is considerably smaller than $J_I^* = 4.11$ m.

5. Conclusions. An interception problem for a switched dynamic system was formulated as a pursuit-evasion differential game with bounded controls. In the case, where the sequences of switch moments and system matrices are known to the pursuer and the evader, the game was solved based on the ZEM in the switched system. As in the case of a nonswitched dynamics, the game space is decomposed into regular and singular zones. In the regular zone, the optimal strategies of the pursuer and the evader are bang-bang depending on the sign of their coefficient functions and the ZEM, and the game value depends on the initial position. In the singular zone, the optimal strategy is arbitrary subject to the constraints, and the game value is constant. If this constant is zero, the singular zone constitutes the robust capture zone.

In the case of incomplete information, the problem was reformulated as two matrix games: the pursuit game (worst case for the pursuer) and the evasion game (worst case for the evader). In the pursuit game, the choice of the switch moments and the dynamics sequence is delegated to the evader, whereas the pursuer assumes the dynamics switches for generating its complete information optimal strategy. In the evasion game, the roles of the adversaries are opposite.

Throughout the paper, the example, where both adversaries have first-order strictly proper dynamics, was considered. In this example, the system dynamics is completely determined by the pair of time constants. For the game with complete information, examples, where a robust capture zone exists and does not exist, were presented. For incomplete information, in all examples, the evasion game had the saddle point. Monte Carlo simulation where the pursuer uses its minimax strategy in the pursuit game and the evader uses its maximin strategy in the evasion game, while the actual switch moments and dynamics sequence are generated randomly, showed that all miss distances fall between the lower value of the evasion game and the upper value of the pursuit game.

In the future, this work can be continued in the following directions.

- In connection with the simulation, where the switch moments and dynamics sequence were generated randomly, a Bayesian game with Nature as the third player can be formulated and investigated.
- The formulation of the pursuit game, where the pursuer's strategy is a feedback one and the evader's extended strategy consists of the switching policy and the feedback, can be investigated as well as a symmetrically formulated evasion game.
- The case where the maximal interval between the dynamics switches tends to zero, creating in the limit a continuous switching, can be strictly formulated and investigated.

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