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Relaxed Dubins Problems through Three Points

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Abstract— **In this paper, we study the Relaxed 3-Point Dubins Problem (R3PDP), which consists of steering a Dubins vehicle through three consecutive waypoints with prescribed heading orientation angle at the initial waypoint. From a geometric point of view, we show that the shortest path must lie in a sufficient family of 12 candidates, and a formula in terms of the parameters of the R3PDP is established for all the 12 candidates. Analyzing the formula indicates that the shortest path of the R3PDP is determined by the zeros of some nonlinear equations. We propose some efficient algorithms to find the zeros of those nonlinear equations so that any R3PDP can be efficiently solved. Finally, some numerical examples are simulated, illustrating the developments of the paper.**

I. Introduction

The Dubins vehicle [6], moving only forward at a constant speed with a minimum turning radius, provides an excellent prototype for nonholonomic vehicles, such as Unmanned Aerial Vehicles (UAVs), ships, unmanned ground vehicles, etc. Hence, its kinematical model has been widely employed in the literature to study the motion planning of those vehicles. Once there are multiple targets to visit by a Dubins vehicle, one needs to solve a Dubins Traveling Salesman Problem (DTSP) [16] which consists of steering a Dubins vehicle to visit each target exactly once and finally return to the initial target so that the path is the shortest.

Solving the DTSP requires not only to order the sequence of waypoints but also to optimize the Heading Orientation Angle (HOA) of the Dubins vehicle when passing through a waypoint. It was proven in [11] that the DTSP is NP-hard. Therefore, solving the DTSP by brute-force optimization is hardly practical when the control decisions have to be made in situ. Due to that difficulty, the Curvature-Constrained Shortest-Path Problem (CCSPP), for which the order of waypoints is fixed in advance, is usually solved to approximate the solution of the DTSP (see [7], [10], [16]).

Since the order of waypoints for CCSPP is fixed, once the HOA at each waypoint is given, the solution path is the concatenation of shortest Dubins paths between two consecutive configurations (a configuration consists of a position and a HOA) according to Bellman's principle of optimality [1]. Recall that the shortest Dubins path between any two configurations can be obtained in a constant time by checking at most six possibilities [6]. As a result, solving the CCSPP amounts to finding a sequence of HOAs at waypoints so that the concatenated path is the shortest. In [7], by partitioning the original CCSPP into some sub-convex problems, a convex programming method was employed to optimize the HOAs; however, this method is hardly practical if the number of waypoints is large. Besides, some approximation-based algorithms have been developed to solve the CCSPP in [10], [14], [16]. An alternating algorithm was proposed in [16] to approximate the HOAs of the CCSPP. Based on the principle of receding horizon, some Look-Ahead Algorithms (LAAs) were developed in [10], [14] to approximate the HOAs of the CCSPP. In [14], the HOA at each waypoint was designed by looking one target ahead along the ordered sequence of waypoints. The 2-step LAA in [10] is a natural extension of [14]. To be specific, the 2-step LAA looks two targets after the current target so that a 3-point Dubins problem (consisting of the current target and the two looked targets) is solved to formulate the HOA at the target subsequent to the current target. This procedure continues until it reaches the final point of the CCSPP.

Unlike the algorithms in [10], [14], [16] to fix the order of waypoints, some algorithms were developed in [4], [5], [9] to solve the DTSP by discretizing the HOA at each waypoint. In [5], the HOA at each waypoint was discretized to formulate an integer optimization programming so that its solution could be computed by an integer programming solver. To reduce the computational complexity of the integer programming, a k-step LAA was proposed in [4], [9] for discretized DTSP inspired by the work of [10].

No matter the sequence of waypoints is fixed or optimized, the 2-step LAA enables keeping an excellent balance between computational complexity and resulting performance (see [4], [5], [9], [10]), in comparison with other aforementioned methods. At each step of the 2-step LAA, it amounts to solving a Relaxed 3-Point Dubins Problem (R3PDP) for which the Dubins vehicle moves through 3 consecutive waypoints with prescribed HOA only at the initial waypoint. Therefore, the computational complexity of the 2-step LAA is not only influenced by the problem's size, i.e., the number of waypoints, but also by the time to solve each R3PDP. In other words, if the time to solve each R3PDP is reduced, the total computational complexity of the 2-step LAA will be accordingly reduced.

Some versions of 3-point Dubins problems were studied in the literature. In [3], [7], [15], the 3-point Dubins problem with fixed headings at both initial and final points were studied. In [13], a 3-point Dubins problem with free headings at all the three points was studied. To out best knowledge, the primary syntheses of the solution of the R3PDP were done in [10].

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Although it was shown in [10] that the solution path of R3PDP must lie in a sufficient family of 12 candidates, the proof for that statement was not complete. In addition, efficient methods to compute the solution of R3PDP were not presented in [10]. In this paper, we provide a complete proof for the statement that the solution path of the R3PDP must lie in a sufficient family of 12 candidates, concatenated by circular arcs (denoted by "C") and straight line segments (denoted by "S"). This result indicates that, once the optimal HOA at the mid point of the R3PDP is given, the shortest path can be computed in a constant time by checking at most 12 possibilities. In order to compute the optimal HOA at the mid point of the R3PDP, we also establish a formula in terms of the parameters of the R3PDP for all the 12 candidates. Analyzing this formula allows to formula a nonlinear equation for each candidate so that the HOA at mid waypoint is a zero of a nonlinear equation. As a consequence, solving the R3PDP is reduced to finding zeros of some nonlinear equations, and we accordingly present an efficient algorithm to solve the R3PDP.

If the distance between any two consecutive waypoints is at least four times of the minimum turning radius, the shortest path is of type CSCS. We observe that, if the shortest path of the R3PDP is of type CSCS, the nonlinear equation can be converted into an 8-degree polynomial. Hence, a standard polynomial solver can be employed to efficiently find the optimal HOA at mid waypoint. In fact, it is reasonable in practical applications to consider that the shortest path of the R3PDP is of type CSCS since any two close targets can be regarded as one waypoint. Numerical simulations (cf. Section V) show that the proposed method is much faster than the straightforward discretization-based method.

II. Preliminary

A. Problem formulation

For a Dubins vehicle that moves only forward at a constant speed with a minimum turning turning radius, its state $\boldsymbol{x} := (x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$, also called configuration, consists of a position vector $(x, y) \in \mathbb{R}^2$ and a HOA $\theta \in \mathbb{S}^1$. Without loss of generality, we assume that the constant speed equals one and that the turning radius is lower bounded by a positive scalar $\rho \in \mathbb{R}_+$. Define two vector fields $\boldsymbol{f}_0$ and $\boldsymbol{f}_1$ on $\mathcal{X} := \mathbb{R}^2 \times \mathbb{S}^1$ by

$$\boldsymbol{f}_0 : \mathcal{X} \to \mathbb{R}^3,\ \boldsymbol{f}_0(\boldsymbol{x}) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix},$$
$$\boldsymbol{f}_1 : \mathcal{X} \to \mathbb{R}^3,\ \boldsymbol{f}_1(\boldsymbol{x}) = \begin{bmatrix} 0 \\ 0 \\ 1/\rho \end{bmatrix}.$$

Then, the kinematics of the Dubins vehicle is given by

$$(\Sigma): \qquad \dot{\boldsymbol{x}}(t) = \boldsymbol{f}_0(\boldsymbol{x}(t)) + u(t)\boldsymbol{f}_1(\boldsymbol{x}(t)),$$

where $t \in \mathbb{R}_+$ denotes the time, the dot denotes the differentiation with respect to time, and $u \in \mathbb{R}$ is the control taking values in $[-1, 1]$.

Throughout the paper, whenever an individual R3PDP is mentioned, we refer to the following definition and notations.

Problem 1 (R3PDP): Given three different waypoints $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, and $\boldsymbol{z}_3$ in $\mathbb{R}^2$, let the HOA at $\boldsymbol{z}_1$ be fixed as $\theta_1 \in [0, 2\pi)$. Then, the R3PDP consists of steering the system (Σ) by a measurable control $u(\cdot) \in [-1, 1]$ on $[0, t_f]$ from $(\boldsymbol{z}_1, \theta_1)$, pathing through $\boldsymbol{z}_2$ at a time $t_2 \in (0, t_f)$, and finally reaching $\boldsymbol{z}_3$ such that the final time $t_f > 0$ is minimized.

As the speed of the Dubins vehicle is a constant, solving the R3PDP is equivalent to finding its shortest path. In addition, we denote by $\theta_2 \in [0, 2\pi]$ hereafter the HOA at $\boldsymbol{z}_2$ along the solution path of the R3PDP.

B. Necessary conditions

Denote by $\boldsymbol{p} = [p_x, p_y, p_\theta] \in T^*_{\boldsymbol{x}}\mathcal{X}$ the costate of $\boldsymbol{x} \in \mathcal{X}$. Then, the Hamiltonian is

$$H(\boldsymbol{x}, \boldsymbol{p}, u) = p_x \cos(\theta) + p_y \sin(\theta) + p_\theta u/\rho.$$

According to the Pontryagin's maximum principle [12], if an admissible controlled trajectory $\boldsymbol{x}(\cdot) = [x(\cdot), y(\cdot), \theta(\cdot)]^T \in \mathcal{X}$ associated with a measurable control $u(\cdot) \in [-1, 1]$ on $[0, t_f]$ ($t_f > 0$) is the solution of the R3PDP, then there exists a $p^0 \leq 0$ and an absolutely continuous mapping $t \mapsto \boldsymbol{p}(\cdot) \in T^*_{\boldsymbol{x}}\mathcal{X}$ on $[0, t_f]$, satisfying $[\boldsymbol{p}(t), p^0] \neq 0$ for $t \in [0, t_f]$, such that, a.e. on $[0, t_f]$, the following equations [Eqs. (1–5)] hold,

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \frac{\partial H(\boldsymbol{x}(t), \boldsymbol{p}(t), u(t))}{\partial \boldsymbol{p}^T}, \\ \dot{\boldsymbol{p}}(t) = -\frac{\partial H(\boldsymbol{x}(t), \boldsymbol{p}(t), u(t))}{\partial \boldsymbol{x}}, \end{cases} \quad \forall t \in [0, t_f] \setminus \{t_2\}, \tag{1}$$

$$H(\boldsymbol{x}(t), \boldsymbol{p}(t), u(t)) = \max_{\eta(t) \in [-1,1]} H(\boldsymbol{x}(t), \boldsymbol{p}(t), \eta(t)), \tag{2}$$

$$-p^0 = H(\boldsymbol{x}(t), \boldsymbol{p}(t), u(t)),\ \forall t \in [0, t_f], \tag{3}$$
$$0 = p_\theta(t_f), \tag{4}$$

and

$$\begin{cases} p_x(t_2^+) = p_x(t_2^-) + \lambda_x, \\ p_y(t_2^+) = p_y(t_2^-) + \lambda_y, \\ p_\theta(t_2^+) = p_\theta(t_2^-), \end{cases} \tag{5}$$

where λ_x and λ_y are scalar constants.

As the abnormal case, i.e., $p^0 = 0$, has been ruled out in [17], the pair $(\boldsymbol{p}, p^0)$ is normalized in this paper such that $p^0 = -1$. Substituting $\boldsymbol{p} = [p_x, p_y, p_\theta]$ explicitly into Eq. (1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p_x(t) \\ p_y(t) \\ p_\theta(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p_x(t)\sin[\theta(t)] - p_y(t)\cos[\theta(t)] \end{bmatrix} \tag{6}$$

This set of equations indicates that $p_x(\cdot)$ and $p_y(\cdot)$ are piecewise constant on $[0, t_f]$. Hence, we have

$$p_\theta = \begin{cases} p_{x_0} y - p_{y_0} x + c_1,\ t < t_2 \\ (p_{x_0} + \lambda_x) y - (p_{y_0} + \lambda_y) x + c_2,\ t > t_2 \end{cases} \tag{7}$$

where p_{x_0} and p_{y_0} are the values of $p_x(\cdot)$ and $p_y(\cdot)$ on $[0, t_2)$, respectively, and c_1 and c_2 are scalar constants.

According to Eq. (7), if $p_\theta(\cdot) \equiv 0$ on a nonzero interval $[\tau_1, \tau_2]$ before or after t_2, the graph of $(x(\cdot), y(\cdot))$ on $[\tau_1, \tau_2]$ forms a straight line segment, indicating $u(\cdot) \equiv 0$ on this interval. Thus, in view of Eq. (2), the switching of u is totally determined by p_θ, i.e.,

$$u = \begin{cases} 1, & p_\theta > 0, \\ 0, & p_\theta \equiv 0, \\ -1, & p_\theta < 0. \end{cases} \tag{8}$$

In the next section, we shall use the above necessary conditions to present some properties for the solution of the R3PDP.

III. Syntheses for the solution of R3PDP

Denote by "S" and "C" a straight line segment and a circular arc with a radius of ρ, respectively. According to [6], [17], the shortest path of a Dubins vehicle between $(\boldsymbol{z}_1, \theta_1)$ and $(\boldsymbol{z}_2, \theta_2)$ belongs to six possibilities in two families:

- CCC={RLR, LRL}, and
- CSC={RSR, RSL, LSL, LSR},

where "R" (resp. "L") means the corresponding circular arc has a right (resp. left) turning direction. Furthermore, according to [2], the shortest path of a Dubins vehicle between $(\boldsymbol{z}_2, \theta_2)$ and $\boldsymbol{z}_3$ belongs to four possibilities in two families:

- CC = {RL,LR}, and
- CS = {RS,LS}.

According to Bellman's principle of optimality [1], the solution path of R3PDP must be the concatenation of the shortest path between $(\boldsymbol{z}_1, \theta_1)$ and $(\boldsymbol{z}_2, \theta_2)$ and the shortest path between $(\boldsymbol{z}_2, \theta_2)$ and $\boldsymbol{z}_3$. Therefore, the solution path of R3PDP must be among the four families:

$$\mathrm{CSC|CS},\ \mathrm{CSC|CC},\ \mathrm{CCC|CS} \text{ and } \mathrm{CCC|CC} \tag{9}$$

where the words before and after "|" denote the path types before and after $\boldsymbol{z}_2$, respectively. It is apparent that the total number of types in Eq. (9) is up to 24. To this end, once the optimal HOA θ_2 at $\boldsymbol{z}_2$ is given, one needs to compare 24 possibilities in order to solve the R3PDP.

A. Types for the solution of R3PDP

By the following lemma, we shall show that the number of types in Eq. (9) can be reduced from 24 to 12.

Lemma 1: Given any R3PDP, if its solution has a type of $\mathrm{C}_1\mathrm{T}_2\mathrm{C}_3|\mathrm{C}_4\mathrm{T}_5$ $(\mathrm{T} \in \{\mathrm{C}, \mathrm{S}\})$ such that none of its subarcs vanishes, we have that C_3 and C_4 have the same turning direction.

Proof: By contradiction, assume that C_3 and C_4 have different turning directions, indicating $p_\theta(t_2) = 0$ at $\boldsymbol{z}_2$.

First of all, we assume $\mathrm{T}_5 = \mathrm{S}$. Note that after t_2 all the points at which $p_\theta = 0$ lie on the same line according to Eq. (7). Since $p_\theta(t_2) = 0$ by assumption, it follows that $\boldsymbol{z}_2$ lies on the same line of T_5, indicating that the length of C_4 is zero, which contradicts with the assumption that none subarcs vanishes. Hence, by contraposition, if $\mathrm{T}_5 = \mathrm{S}$, we have $p_\theta(t_2) \neq 0$, indicating that C_3 and C_4 have the same turning direction.

From now on, let us consider that T_5 is a circular arc. Without loss of generality, we assume $\mathrm{T}_5 = \mathrm{L}$, then the radian $v \in (0, 2\pi)$ of T_5 should be larger than π according to [2]. Since we have $p_\theta = 0$ at the points $\boldsymbol{z}_2$, $\boldsymbol{a}$, and $\boldsymbol{z}_3$, it follows that the three points line on a straight line, as shown in Fig. 1. In this case, there always exists a straight

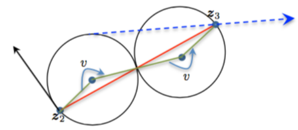

Fig. 1. The geometry of the type RL between $(\boldsymbol{z}_2, \theta_2)$ and $\boldsymbol{z}_3$.

line (the dashed line) tangent C_4 and passing through $\boldsymbol{z}_3$ so that the total length is smaller. In an analogous way, one can prove that there exists a shorter path if $\mathrm{T}_5 = \mathrm{R}$. Hence, if $\mathrm{T}_5 = \mathrm{C}$, the path of $\mathrm{C}_1\mathrm{T}_2\mathrm{C}_3|\mathrm{C}_4\mathrm{T}_5$ is not the shortest. By contraposition, it eventually concludes that C_3 and C_4 have the same turning direction, completing the proof. ■

As a result of this lemma, we immediately have the following result by writing C_3 and C_4 as a single circular arc.

Corollary 1: Given any R3PDP, its solution path must be of a type in

$$\mathcal{F} = \{\mathrm{CSCS}, \mathrm{CSCC}, \mathrm{CCCS}, \mathrm{CCCC}\}$$

or their substrings, where

- $\mathrm{CSCS} = \{\mathrm{RSRS},\ \mathrm{RSLS},\ \mathrm{LSLS},\ \mathrm{LSRS}\}$,
- $\mathrm{CSCC} = \{\mathrm{RSRL},\ \mathrm{RSLR},\ \mathrm{LSLR},\ \mathrm{LSRL}\}$,
- $\mathrm{CCCS} = \{\mathrm{RLRS},\ \mathrm{LRLS}\}$,
- $\mathrm{CCCC} = \{\mathrm{RLRL},\ \mathrm{LRLR}\}$.

To this end, it amounts to checking at most 12 types instead of 24 types in order to solve the R3PDP.

B. Formula for the paths in $\mathcal{F}$

For the sake of notational simplicity, we present some necessary notations that will be used in the remainder of the paper. Let $\theta_i \in [0, 2\pi)$ be the HOA at $\boldsymbol{z}_i$, and set $(x_i, y_i) = \boldsymbol{z}_i$, $i = 1, 2, 3$. Then, we denote by $\boldsymbol{c}_i^r := [x_i + \rho\sin\theta_i, y_i - \rho\cos\theta_i]^T$ and $\boldsymbol{c}_i^l := [x_i - \rho\sin\theta_i, y_i + \rho\cos\theta_i]^T$ the centres of the right and the left circles tangent to the velocity at $\boldsymbol{z}_i$ $(i = 1, 2, 3)$, respectively, as illustrated in Fig. 2.

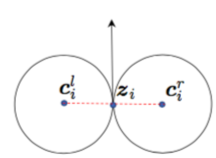

Fig. 2. The geometry for the right center $\boldsymbol{c}_i^r$ and the left center $\boldsymbol{c}_i^l$ associated with $\boldsymbol{z}_i$.

Theorem 1: Given any R3PDP, if its solution path is of a type in $\mathrm{C}_1\mathrm{T}_2\mathrm{C}_3\mathrm{T}_4$ $(\mathrm{T} \in \{\mathrm{S}, \mathrm{C}\})$ such that none of its subarcs vanishes, then the optimal HOA θ_2 at $\boldsymbol{z}_2$ takes a value in $[0, 2\pi)$ such that

$$\frac{\cos(\phi_1 - \theta_2)}{\cos(\alpha_1/2)} = \frac{\cos(\phi_2 - \theta_2)}{\cos(\alpha_2/2)}, \tag{10}$$

where

(i) if $\mathrm{T}_2 = \mathrm{C}$, then $\alpha_1 \in [\pi, 2\pi)$ is the radian of T_2 such that

$$\cos(\alpha_1) = (8\rho^2 - \|\boldsymbol{c}_1^\mu - \boldsymbol{c}_2^\mu\|^2)/8\rho^2 \tag{11}$$

and $\phi_1 \in [0, 2\pi)$ is the orientation angle of the vector $\boldsymbol{c}_2^\mu - \boldsymbol{c}_1^\mu$ where $\mu = r$ if $\mathrm{T}_2 = \mathrm{L}$ and $\mu = l$ otherwise;

(ii) if $\mathrm{T}_4 = \mathrm{C}$, then $\alpha_2 \in (\pi, 2\pi)$ is the radian of T_4 such that

$$\cos(\alpha_2) = (5\rho^2 - \|\boldsymbol{c}_2^\mu - \boldsymbol{z}_3\|^2)/4\rho^2 \tag{12}$$

and $\phi_2 \in [0, 2\pi)$ is the orientation angle of the vector $\boldsymbol{c}_3^\mu - \boldsymbol{c}_2^\mu$ where $\mu = l$ if $\mathrm{T}_4 = \mathrm{R}$ and $\mu = r$ otherwise;

(iii) if $\mathrm{T}_2 = \mathrm{S}$, then $\alpha_1 = 0$ and $\phi_1 \in [0, 2\pi)$ is the orientation angle of the line segment T_2; and

(iv) if $\mathrm{T}_4 = \mathrm{S}$, then $\alpha_2 = 0$ and $\phi_2 \in [0, 2\pi)$ is the orientation angle of the line segment T_4.

The proof of this theorem is elementary based on the geometry of the solution path and interested readers are referred to the proof of [3, Theorem 1]. As a result of Theorem 1, we have that the optimal HOA θ_2 at $\boldsymbol{z}_2$ is a zero of some nonlinear equations, as shown by the following lemmas.

Lemma 2 (CCCC): Given any R3PDP, if its solution path is of type $\mathrm{C}_1\mathrm{C}_2\mathrm{C}_3\mathrm{C}_4$ such that none of its subarcs vanishes, we have

$$\begin{aligned} 0 = {} & 2\frac{[(\cos\theta_2, \sin\theta_2)(\boldsymbol{c}_2^\mu - \boldsymbol{c}_1^\mu)]^2}{\|(\boldsymbol{c}_2^\mu - \boldsymbol{c}_1^\mu)\|^2(16\rho^2 - \|\boldsymbol{c}_1^\mu - \boldsymbol{c}_2^\mu\|^2)} \\ & - \frac{[(\cos\theta_2, \sin\theta_2)(\boldsymbol{c}_3^\mu - \boldsymbol{c}_2^\mu)]^2}{\|\boldsymbol{c}_3^\mu - \boldsymbol{c}_2^\mu\|^2(9\rho^2 - \|\boldsymbol{z}_3 - \boldsymbol{c}_2^\mu\|^2)}, \end{aligned} \tag{13}$$

$$0 = \|\boldsymbol{c}_3^\nu - \boldsymbol{c}_2^\mu\|^2 - 4\rho^2, \tag{14}$$

where $\mu = r$ (resp. l) if $\mathrm{C}_3 = \mathrm{R}$ (resp. L) and $\nu = r$ (resp. l) if $\mathrm{C}_3 = \mathrm{L}$ (resp. R).

Proof: As ϕ_1 is the orientation of $\boldsymbol{c}_2^\mu - \boldsymbol{c}_1^\mu$, it follows that

$$\cos^2(\phi_1 - \theta_2) = \frac{[(\cos\theta_2, \sin\theta_2)(\boldsymbol{c}_2^\mu - \boldsymbol{c}_1^\mu)]^2}{\|\boldsymbol{c}_2^\mu - \boldsymbol{c}_1^\mu\|^2}. \tag{15}$$

Combining $\cos(\alpha) = 2\cos^2(\alpha/2) - 1$ with Eq. (11) leads to

$$\cos^2(\alpha_1/2) = \frac{16\rho^2 - \|\boldsymbol{c}_1^\mu - \boldsymbol{c}_2^\mu\|^2}{16\rho^2}. \tag{16}$$

Analogously, we have

$$\cos^2(\phi_2 - \theta_2) = \frac{[(\cos\theta_2, \sin\theta_2)(\boldsymbol{c}_3^\mu - \boldsymbol{c}_2^\mu)]^2}{\|\boldsymbol{c}_3^\mu - \boldsymbol{c}_2^\mu\|^2}, \tag{17}$$

and combining $\cos(\alpha) = 2\cos^2(\alpha/2) - 1$ with Eq. (12) leads to

$$\cos^2(\alpha_2/2) = \frac{9\rho^2 - \|\boldsymbol{c}_3^\mu - \boldsymbol{c}_2^\mu\|^2}{8\rho^2}. \tag{18}$$

Substituting Eqs. (15–18) into Eq. (10) and squaring the result yield Eq. (13). According to the geometry in Fig. 3, it is apparent that Eq. (14) holds, completing the proof. ■

Lemma 3 (CCCS): Given any R3PDP, if its solution path is of a type $\mathrm{C}_1\mathrm{C}_2\mathrm{C}_3\mathrm{S}_4$ such that none of its subarcs vanishes, we have

$$\begin{aligned} 0 = {} & 16\rho^2\frac{[(\cos\theta_2, \sin\theta_2)(\boldsymbol{c}_2^\mu - \boldsymbol{c}_1^\mu)]^2}{\|(\boldsymbol{c}_2^\mu - \boldsymbol{c}_1^\mu)\|^2(16\rho^2 - \|\boldsymbol{c}_1^\mu - \boldsymbol{c}_2^\mu\|^2)} \\ & - (\cos\phi_2\cos\theta_2 + \sin\phi_2\sin\theta_2)^2, \end{aligned} \tag{19}$$

$$0 = \rho + (\boldsymbol{c}_2^\mu - \boldsymbol{z}_3)^T \begin{pmatrix} \cos(\phi_2 + I_\mu\pi/2) \\ \sin(\phi_2 + I_\mu\pi/2) \end{pmatrix}, \tag{20}$$

where $\mu = r$ (resp. l) if $\mathrm{C}_3 = \mathrm{R}$ (resp. L).

Proof: Substituting Eq. (15) and Eq. (16) into Eq. (10) and squaring the resulting equation lead to Eq. (19). Eq. (20) can be obtained by considering the fact that the vector $(\boldsymbol{c}_2^\mu + \rho(\cos(\phi_2 + I_\mu\pi/2), \sin(\phi_2 + I_\mu\pi/2))^T - \boldsymbol{z}_3)$ is perpendicular with $(\cos(\phi_2 + I_\mu\pi/2), \sin(\phi_2 + I_\mu\pi/2))^T$, completing the proof. ■

Lemma 4 (CSCC): Given any R3PDP, if its solution path is of a type $\mathrm{C}_1\mathrm{S}_2\mathrm{C}_3\mathrm{C}_4$ such that none of its subarcs vanishes, we have

$$\begin{aligned} 0 = {} & [\cos\phi_1\cos\theta_2 + \sin\phi_1\sin\theta_2]^2 \\ & - 8\rho^2\frac{[(\cos\theta_2, \sin\theta_2)(\boldsymbol{c}_3^\mu - \boldsymbol{c}_2^\mu)]^2}{\|\boldsymbol{c}_3^\mu - \boldsymbol{c}_2^\mu)\|^2(9\rho^2 - \|\boldsymbol{c}_3^\mu - \boldsymbol{c}_2^\mu)\|^2)}, \end{aligned} \tag{21}$$

$$0 = \rho(I_\mu - I_\nu)I_\mu - (\boldsymbol{c}_2^\mu - \boldsymbol{c}_1^\nu)^T \begin{pmatrix} \cos(\phi_1 + I_\mu\pi/2) \\ \sin(\phi_1 + I_\mu\pi/2) \end{pmatrix}, \tag{22}$$

$$0 = \|\boldsymbol{c}_3^\omega - \boldsymbol{c}_2^\mu\|^2 - 4\rho^2, \tag{23}$$

where $\mu = r$ (resp. l) if $\mathrm{C}_3 = \mathrm{R}$ (resp. L), $\nu = r$ (resp. l) if $\mathrm{C}_1 = \mathrm{R}$ (resp. L), and $\omega = r$ (resp. l) if $\mathrm{C}_3 = \mathrm{R}$ (resp. L).

Proof: Eq. (21) can be obtained by substituting Eq. (17) and Eq. (18) into Eq. (10). Eq. (22) is a direct result of the fact that $(\cos(\phi_1 + I_\mu\pi/2), \sin(\phi_1 + I_\mu\pi/2))^T$ is perpendicular with $(\boldsymbol{c}_2^\mu + \rho(I_\mu - I_\nu)I_\mu(\cos(\phi_1 + I_\mu\pi/2), \sin(\phi_1 + I_\mu\pi/2))^T - \boldsymbol{c}_1^\nu)$. Eq. (23) can be obtained by the geometry in Fig. 3. ■

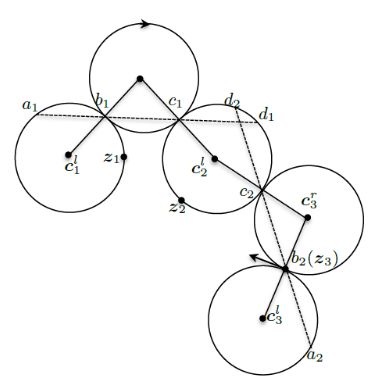

Fig. 3. The geometry of type $LRLR$ for the path of relaxed 3PDP.

Lemma 5 (CSCS): Given any R3PDP, if its solution path is of type $\mathrm{C}_1\mathrm{S}_2\mathrm{C}_3\mathrm{S}_4$ such that none of its subarcs vanishes, and if $\alpha \in [0, 2\pi)$ is half of the radian of C_3, we then have

$$\begin{cases} 0 = (\boldsymbol{c}_2^{\mu} - \boldsymbol{c}_1^{\nu})^T \boldsymbol{e}_1^{\mu} + \rho(I_{\mu} - I_{\nu}) I_{\mu}, \\ 0 = (\boldsymbol{c}_2^{\mu} - \boldsymbol{z}_3)^T \boldsymbol{e}_2^{\mu} + \rho, \end{cases} \tag{24}$$

where $\mu = r$ (resp. l) if $\mathrm{C}_3 = \mathrm{R}$ (resp. L), $\nu = r$ (resp. l) if $\mathrm{C}_1 = \mathrm{R}$ (resp. L), and

$$I_{\mu} = \begin{cases} 1 \text{ if } \mu = r, \\ -1 \text{ if } \mu = l, \end{cases} \quad \boldsymbol{e}_1^{\mu} = I_{\mu} \begin{pmatrix} -\sin(\theta_2 + I_{\mu}\alpha) \\ \cos(\theta_2 + I_{\mu}\alpha) \end{pmatrix},$$

$$\boldsymbol{e}_2^{\mu} = I_{\mu} \begin{pmatrix} -\sin(\theta_2 - I_{\mu}\alpha) \\ \cos(\theta_2 - I_{\mu}\alpha) \end{pmatrix}. \tag{25}$$

The basic idea for proof of this lemma is similar to that of Lemma 4.

IV. ALGORITHMS FOR THE SOLUTION OF R3PDP

In this section, we present efficient algorithms to compute the optimal HOA θ_2 at $\boldsymbol{z}_2$.

A. Discretization-Based Method

A straightforward way to find the optimal HOA θ_2 is to employ a Discretization-Based Method (DBM) (see [4], [5], [9]), which discretizes the HOA θ_2 at $\boldsymbol{z}_2$ over $[0, 2\pi)$ and selects one among the discretized values so that the path is the shortest. Hereafter, we call this method the Straightforward DBM (SDBM) and we use SDBM(l) to denote such a method with a discretization level of $l \in \mathbb{N}$. The details of the SDBM is presented by Algorithm 1.

Algorithm 1 (SDBM): Let $l \in \mathbb{N}$ be the discretization level and set $i = 0$ and $L = \infty$. Then, the SDBM(l) for the R3PDP is performed as follows:

1. If $i < l$, let $\theta = i \times 2\pi/l$ and go to step 2; otherwise, skip to step 3.
2. Let $L_1 \in \mathbb{R}$ be the length of shortest Dubins path between $(\boldsymbol{z}_1, \theta_1)$ and $(\boldsymbol{z}_2, \theta)$ and $L_2 \in \mathbb{R}$ be the length of the shortest Dubins path between $(\boldsymbol{z}_1, \theta)$ and $\boldsymbol{z}_3$;
   - if $L > L_1 + L_2$
     - set $L = L_1 + L_2$, $\theta_2 = \theta$, and $i = i + 1$, and go to step 1;
   - else
     - set $i = i + 1$ and go to step 1.
3. End

The value of θ_2 generated by this algorithm is the best HOA among the discretized angles. Note that the SDBM involves checking the length of Dubins problem for each discretized angle. Next, we shall present a more efficient algorithm by combining the idea of the DBM and the result of Lemmas 2–5.

According to Lemmas 2–5, the variables ϕ_1, ϕ_2, α_1, and α_2 in Eq. (10) are functions of θ_2 and known parameters: $(\boldsymbol{z}_1, \theta_1)$, $\boldsymbol{z}_2$, $\boldsymbol{z}_3$, and ρ. Therefore, the optimal HOA θ_2 at $\boldsymbol{z}_2$ is a zero of the following function

$$F(\theta_2) = \frac{\cos[\phi_1(\theta_2) - \theta_2]}{\cos[\alpha_1(\theta_2)/2]} - \frac{\cos[\phi_2(\theta_2) - \theta_2]}{\cos[\alpha_2(\theta_2)/2]}. \tag{26}$$

To this end, it is enough to solve this nonlinear equation in order to find the optimal HOA θ_2 at $\boldsymbol{z}_2$. However, the nonlinear equation may have multiple zeros and numerical methods may not find all the zeros so that the desired zero may not be found. Below, we present a DBM approximating the zeros of the nonlinear equation in Eq. (26), and we call this method the Nonlinear DBM (NDBM) as shown by Algorithm 2.

Algorithm 2 (NDBM): Let $l \in \mathbb{N}$ be the discretization level and set $i = 1$ and $j = 0$. Then, the NDBM(l) is performed as follows:

1. If $i < l$, let $\theta = i \times 2\pi/l$ and go to step 2; otherwise, go to step 3.
2. Set $F_1 = F((i-1) \times 2\pi/l)$ and $F_2 = F(i \times 2\pi/l)$;
   - if $F_1 \times F_2 < 0$
     - set $j = j+1$, $\theta_2^j = i \times 2\pi/l$, and $i = i+1$, and go to step 1;
   - else
     - set $i = i + 1$ and go to step 1.
3. Select a value among θ_2^j's so that the path of R3PDP is the shortest.

Note that the NDBM(l) checks l times the value of the function in Eq. (26) and only checks a smaller number (depending on the number of zeros of Eq. (26)) of concatenated Dubins paths in order to select the shortest one for the R3PDP. Whereas, the SDBM(l) checks l times concatenated Dubins paths, which is more time consuming than the NDBM(l), as shown by the numerical simulations in Section V.

B. Polynomial-Based Method for CSCS

Observe that the formulas in Lemmas 2–5 are functions of $\sin\theta_2$ and $\cos\theta_2$ if we eliminate other unknown variables, such as ϕ_1, ϕ_2, and θ_3. By considering the half-angle formulas

$$\sin\theta_2 = \frac{2\tan(\theta_2/2)}{1 + \tan^2(\theta_2/2)}, \ \cos\theta_2 = \frac{1 - \tan^2(\theta_2/2)}{1 + \tan^2(\theta_2/2)}, \tag{27}$$

one can transform those formulas into a polynomial in terms of $\tan(\theta_2/2)$ and the coefficients of the resulting polynomial are some combinations of known variables: $(\boldsymbol{z}_1, \theta_1)$, $\boldsymbol{z}_2$, $\boldsymbol{z}_3$, and ρ. (The core idea was presented in [3].) However, the transformation of formulas in Lemmas 2–4 is too complex and the degree of the resulting polynomial is too high to solve by a standard polynomial solver.

In this section, we shall only consider Eq. (24) which can be converted into an 8-degree polynomial, as shown by [3, Lemma 3]. As a consequence, if the solution path is of type CSCS, one obtains the value of $\tan(\theta_2/2)$ by finding the zeros of the 8-degree polynomial. (A standard polynomial solver can be employed to efficiently find all the zeros of the polynomial, as shown by the numerical simulations in Section V.) We present below the details of the Polynomial-Based Method (PBM) to compute the HOA θ_2 at $\boldsymbol{z}_2$.

Algorithm 3 (PBM): If the solution path of R3PDP is of type CSCS, we perform the PBM by the following steps:

1. Find all the roots $r_i \in \mathbb{C}$ $(i = 1, \cdots, 8)$ of the 8-degree polynomial by a standard polynomial solver, e.g., the subroutine of "roots" in MATLAB.

2. Let $k > 0$ be the number of real roots and denote by $\hat{r}_j \in \mathbb{R}$ $(j = 1, \cdots, k)$ the real roots among r_i's.
3. Set $\hat{\theta}_j = 2\arctan(\hat{r}_j)$ for $j = 1, \cdots, k$.
4. Select a value among $\hat{\theta}_j$'s so that the path of type $CSCS$ is the shortest.

With this procedure, the optimal HOA θ_2 at $\boldsymbol{z}_2$ can be efficiently computed if the path is of type CSCS. In the next section, we shall present some numerical simulations to show the performance of the PBM.

V. Numerical simulations

A. Performance of NDBM

Set $(\boldsymbol{z}_1, \theta_1) = (0, 0, \pi/2)$ and $\rho = 1$. Let $\boldsymbol{z}_2$ and $\boldsymbol{z}_3$ be generated randomly by uniform distribution. Then, both the NDBM(360) and SDBM(360) are tested on 10000 randomly generated R3PDPs. The average improvement factors of the NDBM(360) in terms of time consumption compared with the SDBM(360) are presented in Tab. I.

According to Algorithms 1 and 2, while the SDBM(360) checks 360 times Dubins paths to select the shortest path for the R3PDP, the NDBM(360) just needs to check 360 times the value of the function in Eq. (26) and to check smaller times the Dubins paths depending on the number of zeros of the nonlinear equations in Lemmas 2–5. (Numerical results show that the average number of zeros for the nonlinear equations for each type in $\mathcal{F}$ is around 8.) As checking the value of a nonlinear function in Eq. (26) is less time consuming than checking the length of a Dubins path, it is reasonable to see that the NDBM(360) is less time consuming, as shown by the improvement factors in Tab. I.

TABLE I

The average improvement factors of the NDBM(360) in terms of time consumption compared with the SDBM(360).

Path Types	CCCC	CCCS	CSCC	CSCS
Improvement Factor	3.392	4.818	4.753	4.935

B. Performance of PBM for CSCS

Set $(\boldsymbol{z}_1, \theta_1) = (0, 0, \pi/2)$ and $\rho = 1$. Let the two points $\boldsymbol{z}_2$ and $\boldsymbol{z}_3$ be generated randomly by uniform distribution. Both the SDBM(360) and the PBM are tested on 10000 randomly generated R3PDPs. Note that performing the PBM requires solving an 8-degree polynomial. Thanks to the development of QR algorithm for finding matrix eigenvalues, each polynomial can be solved efficiently in general [8, p. 94]. In this simulation, the subroutine of "roots" in MATLAB is employed to finding the zeros of the polynomial.

The numerical result shows that the improvement factor of the PBM is up to 39.63 in comparison with SDBM(360). Such a large improvement factor is reasonable because the PBM only involves solving an 8-degree polynomial and checking at most 8 HOAs (an 8-degree polynomial has at most 8 real roots). However, the SDBM(360) checks 360 HOAs for each type of CSCS.

VI. Conclusions

The shortest path of the R3PDP is synthesized. It is shown that the shortest path of the R3PDP must be among 12 types or their substrings. So, at most 12 possibilities are needed to check in order to solve each R3PDP. A formula for all the 12 types is established, revealing that the optimal HOA at the mid point of the R3PDP is a zero of some nonlinear equations. In order to find the optimal HOA at the mid point, we propose a discretization-based method to approximate the zeros of nonlinear equations, as shown by the simulations in Sect. V this method is more efficient than the straightforward discretization-based method used in the literature. Furthermore, if the solution path is of type CSCS, the optimal HOA at the mid point of the R3PDP can be more efficiently obtained by finding the zeros of an eight-degree polynomial.

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