JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 41, No. 4, April 2018

Necessary Conditions for “Hit-to-Kill” in Missile Interception Engagements

Shmuel Y. Hayoun* and Tal Shima†
Technion–Israel Institute of Technology, 32000 Haifa, Israel

DOI: 10.2514/1.G003052

The endgame of a linearized endoatmospheric interception scenario between an evading target and a pursuing missile is considered, in which the adversaries are aerodynamically steered and their controls are bounded and have arbitrary-order dynamics. The common near head-on or tail-chase assumption is relaxed and a new condition by which the dimension of the kinematics may be reduced is obtained. Assuming perfect information, the necessary and sufficient condition for the existence of a “hit-to-kill” capture zone is presented. The existence of such a capture zone is a necessary condition for guaranteeing point capture against any target maneuver. The condition is expressed as a function of the adversaries' arbitrary control dynamics, and explicit conditions are derived for several previously studied cases, complimenting known results.

I. Introduction

A MISSILE designed for the interception of a maneuvering target is commonly fitted with a warhead. This guarantees (with some probability) the destruction of the designated target for a range of miss distances, giving the missile a nonzero kill radius. However, in high-speed interception engagements, such as defense against ballistic missiles, the effectiveness of the warhead is reduced considerably, raising the need for “hit-to-kill.” It is therefore desired in such scenarios to be able to ensure the interceptor's capability to impact with the target. In an age of intelligent evasive targets this capability to exactly capture the target (capturability) is difficult to guarantee and is dependent first and foremost on the maneuver capabilities and dynamic response of the pursuing missile.

The first work to specifically address the conditions for capturability of an evading target was presented by Cockayne [1]. It considered the so-called game of two cars [2]: a 1-on-1 planar engagement with nonlinear motion in which both adversaries have constant speeds, bounded lateral accelerations (path curvature constraints), and ideal control dynamics, and each adversary's current and future maneuvers are unknown to its opponent (only their position and attitude are known). It was proved that the pursuer can capture the evader (achieve position coincidence) from any initial state if and only if it has a speed advantage and is at least as maneuverable as the evader. This theory was later extended in [3] to address motion in three-dimensional space. It was shown that a sufficient condition for capturability is the pursuer's superiority both in speed and in maneuverability. In [4] an inverse study to Cockayne's was presented. It was proved that in the game of two cars the evader can avoid capture for any initial conditions if and only if one of the following holds: 1) it has a speed advantage and its maximal maneuver capability is greater than or equal to that of the pursuer times the pursuer-to-evader speed ratio; 2) its speed is equal to the pursuer's and it has a maneuverability advantage. Based on these results and those presented by Cockayne it was also deduced that if the pursuer has a speed advantage but its maximal maneuver capability is lesser than that of the evader times the evader-to-pursuer speed ratio, then there exist initial conditions from which it can guarantee the evader's capture. Preceding these publications was Isaacs' study of pursuit-evasion games [2]. The optimality of the evading target's strategy in such games suggests that, in the same framework, any capturability analysis results must agree with conditions for the existence of capture regions obtained in the game solution. Accordingly, Cockayne claimed in [1] that his capture conditions should coincide with Isaacs' results in the game of two cars with zero capture radius. In a recent publication [5] Gutman and Rubinsky introduced the differential-game-based “vector guidance” for an accelerating missile in an exo-atmospheric interception engagement. Under the assumption of ideal adversaries and a greater acceleration bound of the pursuer, they derived a first-pass capturability condition as a function of the initial relative geometry. Any initial conditions from which first-pass capture is impossible are such that second-pass capture is necessarily possible, as a result of the pursuer's maximum acceleration advantage.

In scenarios where during the endgame the adversaries' motion is near their respective collision courses, the kinematics of the engagement can be linearized relative to some fixed frame [6]. The capturability in such cases is comparable to capture zone existence conditions derived from the solution of linear games of pursuit. Existing solutions to linear pursuit-evasion games of a single pursuer versus a single evader with bounded controls also include variations on the order of the players' control dynamics. In [7] the solution to a simplified linear pursuit-evasion game in which both players have ideal control dynamics was given. A general solution for an arbitrary set of linear system dynamics was later presented in [8], which also included an analysis of the specific case of a pursuer with first-order control dynamics intercepting an ideal evader. This was followed by [9], in which a game with both players having first-order strictly proper control dynamics was analyzed. Later on it was shown that in the same framework point capture is possible if and only if 1) the pursuer does not have a maneuverability or an agility disadvantage or 2) the pursuer is only more agile [10,11]. In [12] an analysis of a class of linear time-varying feedback pursuit strategies in the same framework was presented, focusing on scenarios in which point capture is guaranteed. A further extension of the differential-game–based solutions was presented in [13], which considered a conflict between a pursuer with biproper control dynamics and an ideal target. Later on, in [14], an analysis yielding some of the necessary conditions for capturability in a two-player game in which both adversaries have biproper control dynamics were presented. Further interesting results were obtained in [15,16] for the case of a dual-controlled missile with biproper dynamics intercepting an evader with first-order control dynamics and for the case of a missile with second-order control dynamics versus an evader with first-order control dynamics, respectively. While these previous studies assumed perfect information, [17] dealt with the required estimation capabilities of a more maneuverable pursuer in order to ensure capture.

These previous studies have yielded important conclusions with regard to the necessary requirements from interceptors in

Received 11 May 2017; revision received 12 July 2017; accepted for publication 14 July 2017; published online 30 August 2017.

*Graduate Student, Department of Aerospace Engineering; shmuli@tx.technion.ac.il.

†Associate Professor, Department of Aerospace Engineering; tal.shima@technion.ac.il. Associate Fellow AIAA.

engagements where point capture is desired. Additionally, the linear analyses also considered the influence of the adversaries' internal dynamics on their performance, which was previously unaccounted for. However, their validity is restricted only to scenarios in which the relative geometry is near head-on or tail-chase. Furthermore, the currently existing complete capturability conditions encompass a fairly limited set of specific simple low-order dynamics cases. From a practical point of view these issues are of considerable importance, because in reality the relative geometry in interception scenarios may be far from head-on or tail-chase and the dynamics of aerial vehicles may be of arbitrary order. Moreover, as was illustrated by Rusnak and Meir [18], guidance laws that use low-order approximations of high-order dynamics yield inferior performance relative to guidance laws that take into account the complete high-order autopilot. Similarly, any inaccuracies in the dynamic model may lead to wrong conclusions with regard to capturability. Following the presented works, it is of interest to examine the necessary conditions for capture in a wider class of interception scenarios, in which the nominal collision triangle's geometry may be far from head-on or tail-chase, and that include a more generalized representation of the adversaries' autopilots. In addition, the capability to represent these conditions in terms of the adversaries' dynamics parameters is of significant practical importance.

This paper presents an analytical study of the necessary conditions for the feasibility of exact capture in a linearized interception engagement in which the adversaries' control dynamics may be of arbitrary order and the relative geometry during the endgame may be far from head-on or tail-chase. First, a generalization of the conditions under which the dimensions of the linearized kinematics may be reduced is performed. Then a general necessary condition, based on the solution of a perfect information linear differential game of pursuit, is presented. Later on it is expressed in terms of the adversaries' control dynamics parameters, generalizing the currently existing conditions for only some specific cases of the adversaries' dynamics. Several examples of previously studied cases are then given, for which the capture conditions are expressed explicitly and compared with the known results. Identical results are obtained for the cases of the adversaries both having either ideal or first-order strictly proper control dynamics. In the cases of the adversaries having either first-order biproper or second-order strictly proper control dynamics and dual-controlled adversaries with first-order biproper control dynamics, the currently existing conditions are extended and presented in full. Finally, some numerical results are shown for the case of dual-controlled adversaries with first-order biproper control dynamics, validating the obtained existence conditions.

The remainder of the paper is arranged as follows: In Sec. II a mathematical model of the interception engagement is presented. The problem is then formulated as a differential game, the solution of which is outlined in Secs. III and IV includes the derivation of general test by which to determine the existence of a capture zone, as well as the analysis which enables this test to be expressed in terms of the adversaries'dynamic parameters. Several examples are given in Sec. V, followed by numerical simulations in Sec. VI and concluding remarks in Sec. VII.

II. Engagement Formulation

Consider the endgame geometry of a planar endoatmospheric interception engagement between two aerodynamically steered adversaries in some fixed Cartesian inertial frame X–Y, as shown in Fig. 1. V and a denote the speed and lateral acceleration, respectively, and γ and λ (positive in the counterclockwise direction), respectively, denote the path and line-of-sight (LOS) angles from X. Subscript P denotes the pursuer and E the evader. Subscript/superscript o denotes the value at the initial time t_o. γ_j^{col} denotes the initial collision path angle of j, where

$$\gamma_E^{\text{col}} = \gamma_E^o \tag{1}$$

and, by definition of the initial collision triangle, γ_P^{col} is defined as the angle maintaining

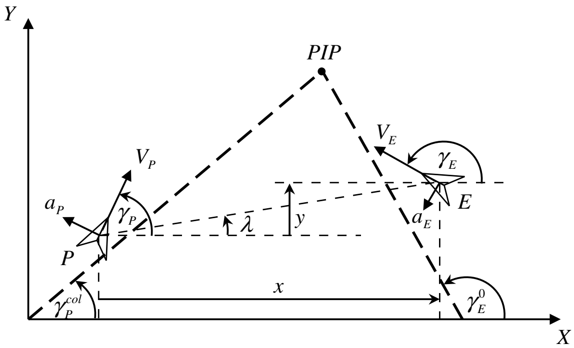

Fig. 1 Two-on-one engagement scheme.

$$V_P \sin(\gamma_P^{\text{col}} - \lambda_o) = V_E \sin(\gamma_E^{\text{col}} - \lambda_o) \tag{2}$$

The nonlinear kinematics are given by

$$\begin{aligned} \dot{x} &= V_E \cos(\gamma_E - \lambda_o) - V_P \cos(\gamma_P - \lambda_o) \\ \dot{y} &= V_E \sin(\gamma_E - \lambda_o) - V_P \sin(\gamma_P - \lambda_o) \end{aligned} \tag{3}$$

where x and y are the displacements of P, respectively, along and normal to its initial LOS, relative to E. The evolution of the path angles is according to

$$\dot{\gamma}_j = \frac{a_j}{V_j}, \qquad j \in \{P, E\} \tag{4}$$

Assuming, for simplicity, that the X axis is fixed along the initial LOS ($\lambda_o = 0$), Eq. (3) becomes

$$\begin{aligned} \dot{x} &= V_E \cos(\gamma_E) - V_P \cos(\gamma_P) \\ \dot{y} &= V_E \sin(\gamma_E) - V_P \sin(\gamma_P) \end{aligned} \tag{5}$$

and Eq. (2) becomes

$$V_P \sin(\gamma_P^{\text{col}}) = V_E \sin(\gamma_E^{\text{col}}) \tag{6}$$

We also assume the following:

1) The adversaries' speeds are constant during the endgame phase and their control inputs are bounded.

2) The adversaries can be represented by point masses with linear control dynamics.

3) The adversaries' trajectories can be linearized around their respective collision paths ($\Delta\gamma_j \triangleq \gamma_j - \gamma_j^{\text{col}} \ll 1$, $j \in \{P, E\}$).

4) A perfect information structure exists (both adversaries have complete knowledge of the states at all times).

Remark II.1: The third assumption is in fact practical in high-speed interception engagements where the maneuver bounds of the adversaries are small compared with their respective speeds; that is, their minimum turn radii are much larger than the distance between them.

A. Kinematics Model

Under the aforementioned assumptions

$$\begin{aligned} \sin(\gamma_j) &= \sin(\gamma_j^{\text{col}} + \Delta\gamma_j) \approx \sin(\gamma_j^{\text{col}}) + \cos(\gamma_j^{\text{col}}) \cdot \Delta\gamma_j \\ \cos(\gamma_j) &= \cos(\gamma_j^{\text{col}} + \Delta\gamma_j) \approx \cos(\gamma_j^{\text{col}}) - \sin(\gamma_j^{\text{col}}) \cdot \Delta\gamma_j \end{aligned} \tag{7}$$

and the following linearized kinematics of the relative intercept geometry are obtained:

$$\begin{aligned} \dot{x} &= V_E\Big[\cos(\gamma_E^{\text{col}}) - \sin(\gamma_E^{\text{col}}) \cdot \Delta\gamma_E\Big] - V_P\Big[\cos(\gamma_P^{\text{col}}) - \sin(\gamma_P^{\text{col}}) \cdot \Delta\gamma_P\Big] \\ \dot{y} &= V_E\Big[\sin(\gamma_E^{\text{col}}) + \cos(\gamma_E^{\text{col}}) \cdot \Delta\gamma_E\Big] - V_P\Big[\sin(\gamma_P^{\text{col}}) + \cos(\gamma_P^{\text{col}}) \cdot \Delta\gamma_P\Big] \end{aligned} \tag{8}$$

where the dynamics of $\Delta\gamma_j$, $j \in \{P, E\}$ are given in Eq. (4). Substituting Eq. (6) and the closing speed between P and E on the collision triangle along X, $V_c^{\text{col}} = V_P \cos(\gamma_P^{\text{col}}) - V_E \cos(\gamma_E^{\text{col}})$, yields

$$\dot{x} = -V_c^{\mathrm{col}} + V_P \sin(\gamma_P^{\mathrm{col}}) \cdot \Delta\gamma_P - V_E \sin(\gamma_E^{\mathrm{col}}) \cdot \Delta\gamma_E$$
$$\dot{y} = V_E \cos(\gamma_E^{\mathrm{col}}) \cdot \Delta\gamma_E - V_P \cos(\gamma_P^{\mathrm{col}}) \cdot \Delta\gamma_P \tag{9}$$

In many previous works $\dot{x}$ was eliminated from the kinematics by assuming that the interception scenario is close to either head-on or tail-chase [6,7,9,19]. By examining Eq. (9) it is evident that in order to eliminate $\dot{x}$ we must actually assume that either

$$|\Delta\gamma_j| \ll |\cot(\gamma_j^{\mathrm{col}})|, \qquad j \in \{P, E\} \tag{10}$$

in which case $|\sin(\gamma_j^{\mathrm{col}}) \cdot \Delta\gamma_j| \ll |\cos(\gamma_j^{\mathrm{col}})|$, $j \in \{P, E\}$, or

$$\Delta\gamma_P \approx \Delta\gamma_E \tag{11}$$

in which case, following Eq. (6), $V_P \sin(\gamma_P^{\mathrm{col}}) \cdot \Delta\gamma_P - V_E \sin(\gamma_E^{\mathrm{col}}) \cdot \Delta\gamma_E \approx 0$.

Whereas Eq. (11) might in many scenarios be an inappropriate assumption, because in essence it dictates the behavior of one adversary based on its opponents maneuvers, Eq. (10) poses a more lenient restriction on the relative engagement geometry, relative to the previously common near head-on or tail-chase assumption. This condition allows us to consider a wider range of linear interception scenarios in which the collision triangle is far from head-on or tail-chase. Note that for the specific cases of near head-on ($\gamma_P^{\mathrm{col}} \approx 0°$, $\gamma_E^{\mathrm{col}} \approx 180°$), near tail-chase ($\gamma_P^{\mathrm{col}} \approx \gamma_E^{\mathrm{col}} \approx 0°$), or near head pursuit ($\gamma_P^{\mathrm{col}} \approx \gamma_E^{\mathrm{col}} \approx 180°$) Eq. (10) naturally holds (since $|\Delta\gamma_j| \ll 1$ and $|\cot(\gamma_j)| \to \infty \ \forall j \in \{P, E\}$).

Assuming that Eq. (10) holds, Eq. (9) becomes

$$\dot{x} = -V_c^{\mathrm{col}}$$
$$\dot{y} = V_E \cos(\gamma_E^{\mathrm{col}}) \cdot \Delta\gamma_E - V_P \cos(\gamma_P^{\mathrm{col}}) \cdot \Delta\gamma_P \tag{12}$$

The boundary conditions include the given initial values t_o, $x(t_o) = x_o$, $y(t_o) = 0$, $\Delta\gamma_P(t_o) = \gamma_P^o - \gamma_P^{\mathrm{col}}$, and $\Delta\gamma_E(t_o) = 0$, as well as the terminal condition $x(t_f) = 0$, where t_f, the interception time, is defined as the moment E passes P ($x = 0$). By differentiating $\dot{y}$ and substituting Eq. (4) we obtain the following well-known linear kinematics:

$$\dot{y} = v$$
$$\dot{v} = a_E \cos(\gamma_E^{\mathrm{col}}) - a_P \cos(\gamma_P^{\mathrm{col}}) \tag{13}$$

Integration of the equation for $\dot{x}$ from t_o to t_f yields the familiar approximated interception time:

$$t_f = t_o + \frac{x_o}{V_c^{\mathrm{col}}} \tag{14}$$

We define the time-to-go as

$$t_{\mathrm{go}} = t_f - t \tag{15}$$

B. Control Dynamics Models

Assuming linear arbitrary-order control dynamics of the adversaries

$$\begin{aligned} \dot{\boldsymbol{\zeta}}_j &= \boldsymbol{A}_\zeta^j \boldsymbol{\zeta}_j + \boldsymbol{B}_\zeta^j \boldsymbol{u}_j \\ a_j &= \boldsymbol{c}_\zeta^j \boldsymbol{\zeta}_j + \boldsymbol{d}_\zeta^j \boldsymbol{u}_j \end{aligned}, \qquad j \in \{P, E\} \tag{16}$$

where $\boldsymbol{A}_\zeta^j \in \mathbb{R}^{n_\zeta^j \times n_\zeta^j}$, $\boldsymbol{B}_\zeta^j \in \mathbb{R}^{n_\zeta^j \times n_c^j}$, $\boldsymbol{c}_\zeta^j \in \mathbb{R}^{1 \times n_\zeta^j}$, and $\boldsymbol{d}_\zeta^j \in \mathbb{R}^{1 \times n_c^j}$. $\boldsymbol{\zeta}_j$ is j's vector of n_ζ^j internal dynamic states and $\boldsymbol{u}_j$ is j's vector of n_c^j dimensionless control inputs, each of which is bounded by

$$|u_i^j| \le \bar{u}_i^j, \qquad i \in N_c^j$$

By defining

$$u_j^{\max} = \sum_{i=1}^{n_c^j} \bar{u}_i^j \tag{17}$$

we may rewrite

$$\begin{aligned} |u_i^j| &\le \gamma_i^j \cdot u_j^{\max} \\ 0 &\le \gamma_i^j \le 1 \end{aligned}, \qquad i \in N_c^j \tag{18}$$

where, by definition,

$$\gamma_i^j = \frac{\bar{u}_i^j}{u_j^{\max}} \tag{19}$$

and each $\boldsymbol{\Gamma}_j = [\gamma_1^j \quad \gamma_2^j \quad \ldots \quad \gamma_{n_c^j}^j]^T$ satisfies

$$\|\boldsymbol{\Gamma}_j\|_1 = \sum_{i=1}^{n_c^j} \gamma_i^j = 1, \qquad j \in \{P, E\}$$

In essence, each element in $\boldsymbol{\Gamma}_j$ represents the relative level of effectiveness of its corresponding control input.

C. Linear Engagement Model

Defining the following state vector

$$\boldsymbol{\xi}(t) = \begin{bmatrix} y(t) & v(t) & \boldsymbol{\zeta}_P(t)^T & \boldsymbol{\zeta}_E(t)^T \end{bmatrix}^T, \qquad j \in \{P, E\} \tag{20}$$

results in the following linear time-invariant system:

$$\dot{\boldsymbol{\xi}} = \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{B}_P \boldsymbol{u}_P + \boldsymbol{B}_E \boldsymbol{u}_E; \quad \boldsymbol{\xi}(t_o) = \boldsymbol{\xi}_o, \quad \boldsymbol{u}_j \in \boldsymbol{U}_j, \quad j \in \{P, E\} \tag{21}$$

where $\boldsymbol{U}_j = \{\boldsymbol{v} | \boldsymbol{v} \in \mathbb{R}^{n_c^j}, |v_i| \le \gamma_i^j \cdot u_j^{\max} \ \forall i \in N_c^j\}$ is j's set of admissible controls and

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & \boldsymbol{0}_{1 \times n_\zeta^P} & \boldsymbol{0}_{1 \times n_\zeta^E} \\ 0 & 0 & -\boldsymbol{c}_\zeta^P \cos(\gamma_P^{\mathrm{col}}) & \boldsymbol{c}_\zeta^E \cos(\gamma_E^{\mathrm{col}}) \\ \boldsymbol{0}_{n_\zeta^P \times 1} & \boldsymbol{0}_{n_\zeta^P \times 1} & \boldsymbol{A}_\zeta^P & \boldsymbol{0}_{n_\zeta^P \times n_\zeta^E} \\ \boldsymbol{0}_{n_\zeta^E \times 1} & \boldsymbol{0}_{n_\zeta^E \times 1} & \boldsymbol{0}_{n_\zeta^E \times n_\zeta^P} & \boldsymbol{A}_\zeta^E \end{bmatrix},$$

$$\boldsymbol{B}_P = \begin{bmatrix} 0 \\ -\boldsymbol{d}_P \cos(\gamma_P^{\mathrm{col}}) \\ \boldsymbol{B}_\zeta^P \\ \boldsymbol{0}_{n_\zeta^E \times 1} \end{bmatrix}, \quad \boldsymbol{B}_E = \begin{bmatrix} 0 \\ \boldsymbol{d}_E \cos(\gamma_E^{\mathrm{col}}) \\ \boldsymbol{0}_{n_\zeta^P \times 1} \\ \boldsymbol{B}_\zeta^E \end{bmatrix} \tag{22}$$

$\boldsymbol{0}_{p \times q}$ denoting a zero matrix of dimensions $p \times q$ and n_ζ^j the number of elements in $\boldsymbol{\zeta}_j$.

D. Order Reduction

Using the terminal projection transformation [20] we reduce the order of the linear system in Eq. (21) to a single scalar known as the zero effort miss (ZEM):

$$z(t) = \boldsymbol{D}\boldsymbol{\Phi}(t_f, t)\boldsymbol{\xi}(t) \tag{23}$$

where

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & \boldsymbol{0}_{1 \times n_z^P} & \boldsymbol{0}_{1 \times n_z^E} \end{bmatrix} \tag{24}$$

and $\boldsymbol{\Phi}$ is the transition matrix associated with Eq. (21). The resulting reduced order dynamic equations are

$$\dot{z}(t) = \boldsymbol{D}[\dot{\boldsymbol{\Phi}}(t_f, t)\boldsymbol{\xi} + \Phi(t_f, t)\dot{\boldsymbol{\xi}}] = \boldsymbol{D}\Phi(t_f, t)[\boldsymbol{B}_P\boldsymbol{u}_P + \boldsymbol{B}_E\boldsymbol{u}_E];$$
$$z(t_o) = \boldsymbol{D}\Phi(t_f, t)\boldsymbol{\xi}_o = \boldsymbol{z}_o \tag{25}$$

Defining

$$\boldsymbol{f}_P(t_f, t) \triangleq \boldsymbol{D}\Phi(t_f, t)\boldsymbol{B}_P$$
$$\boldsymbol{f}_E(t_f, t) \triangleq \boldsymbol{D}\Phi(t_f, t)\boldsymbol{B}_E \tag{26}$$

yields

$$\dot{z} = \boldsymbol{f}_P(t_f, t)\boldsymbol{u}_P + \boldsymbol{f}_E(t_f, t)\boldsymbol{u}_E; \qquad z(t_o) = z_o \tag{27}$$

III. Differential Game Solution

Assuming that the evader's control input is unknown to the pursuer throughout the engagement, an optimal guidance algorithm can be obtained from the solution of a linear pursuit-evasion game. The cost function of the game is chosen to be

$$J = |y(t_f)| \tag{28}$$

The game cost in terms of the ZEM variables is, by definition,

$$J = |z(t_f)| \tag{29}$$

The Hamiltonian in this two-sided optimization problem is

$$H = \lambda_z\dot{z} = \lambda_z\Big[\boldsymbol{f}_P(t_f, t)\boldsymbol{u}_P + \boldsymbol{f}_E(t_f, t)\boldsymbol{u}_E\Big] \tag{30}$$

The optimal strategies must satisfy

$$\boldsymbol{u}_P^* = \underset{\boldsymbol{u}_P\in\boldsymbol{U}_P}{\arg\min} H = -\mathrm{sign}\{\lambda_z\boldsymbol{f}_P(t_f, t)\}\cdot\boldsymbol{\Gamma}_P\cdot u_P^{\max}$$
$$\boldsymbol{u}_E^* = \underset{\boldsymbol{u}_E\in\boldsymbol{U}_E}{\arg\max} H = \mathrm{sign}\{\lambda_z\boldsymbol{f}_E(t_f, t)\}\cdot\boldsymbol{\Gamma}_E\cdot u_E^{\max} \tag{31}$$

The adjoint equations and transversality conditions are

$$\frac{\mathrm{d}\lambda_z}{\mathrm{d}t} = -\frac{\partial H}{\partial z} = 0; \quad \lambda(t_f) = \frac{\partial J}{\partial z}\bigg|_{t=t_f} = \mathrm{sign}\{z(t_f)\}, \quad z(t_f) \neq 0 \tag{32}$$

Hence, as long as λ is continuous

$$\lambda = \mathrm{sign}\{z(t_f)\} \tag{33}$$

and

$$\boldsymbol{u}_P^* = -\mathrm{sign}\{z(t_f)\}\cdot\mathrm{sign}\{\boldsymbol{f}_P(t_f, t)\}\cdot\boldsymbol{\Gamma}_P\cdot u_P^{\max}$$
$$\boldsymbol{u}_E^* = \mathrm{sign}\{z(t_f)\}\cdot\mathrm{sign}\{\boldsymbol{f}_E(t_f, t)\}\cdot\boldsymbol{\Gamma}_E\cdot u_E^{\max} \tag{34}$$

Substituting these open-loop optimal controls in Eq. (27) and integrating from t to t_f yields the candidate optimal trajectories

$$z^*(t) = z(t_f) - \mathrm{sign}\{z(t_f)\}\cdot\int_t^{t_f}[F_E(t_f,\theta)\cdot u_E^{\max} - F_P(t_f,\theta)\cdot u_P^{\max}]\mathrm{d}\theta \tag{35}$$

where

$$F_j(t_f, t) = \sum_{i=1}^{n_c^j}\gamma_i^j\cdot\big|f_i^j(t_f, t)\big|, \qquad j\in\{P, E\} \tag{36}$$

IV. Capture Zone Existence Conditions

We continue our analysis with the objective of expressing the conditions for the existence of a capture zone explicitly, in terms of the dynamic constants of the adversaries. The importance of such conditions is clearer from a "negative" point of view; that is, if in a given scenario the dynamic characteristics of the pursuer, relative to those of the evader, do not maintain the necessary and sufficient conditions, then, in the present framework, it is clear that the pursuer cannot guarantee the evader's capture, regardless of the initial conditions. If, however, these conditions do hold, then, obviously, both the structure of the capture region and the optimal pursuer's strategy, guaranteeing the capture from this region, are of interest. In such a case both can be provided by the solution of a linear pursuit-evasion game; the optimal controls are as derived in [8] and the construction of the capture zone, if such exists, can be found in [21]. Unlike [5] or [17], further considerations, such as defining the specific initial conditions that compose the capture zone or the negative effect of estimation on capturability, are beyond the scope of this paper.

Definition IV.1 (capture zone): The nonempty set of all initial conditions from which the pursuer is capable of guaranteeing point capture is called the capture zone.

$$\{(t_o, z_o)|\boldsymbol{u}_P = \boldsymbol{u}_P^*, \boldsymbol{u}_E\in\boldsymbol{U}_E{:}z(t_f)0\}$$

Lemma IV.1: The necessary and sufficient condition for the existence of a capture zone in a linear 1-on-1 engagement is

$$\exists t_o < t_f\colon u_P^{\max}F_P(t_f, t) - u_E^{\max}F_E(t_f, t) \geq 0 \quad \forall\, t\in[t_o, t_f] \tag{37}$$

Proof: Follows the same logic as the proof of Theorem 4.1 in [14]. □

Remark IV.1: This is in fact a generalization for a multi-input case of known conditions for the existence of a capture zone in differential games of pursuit with bounded controls [14,21]. Actually, the left-hand side of Eq. (37) is equal to the negative of the so-called determining function defined in [21].

Because Eq. (37) is necessary and sufficient for some $t_o > 0$, then as long as its value does not reach or exceed the terminal instant, t_o may be arbitrarily large. A valid test to examine whether Eq. (37) holds is therefore

$$\lim_{t\to t_f^-}\Big[u_P^{\max}F_P(t_f, t) - u_E^{\max}F_E(t_f, t)\Big] \geq 0 \tag{38}$$

From Eq. (16) the control dynamics can be represented by the following vector of transfer functions from the control inputs vector to the lateral acceleration:

$$\boldsymbol{H}_j(s) = \Big[\, H_1^j(s) \quad H_2^j(s) \quad \dots \quad H_{n_c^j}^j(s)\,\Big]^T$$
$$\triangleq \boldsymbol{c}_\zeta^j(s\boldsymbol{I} - \boldsymbol{A}_\zeta^j)^{-1}\boldsymbol{B}_\zeta^j + \boldsymbol{d}_\zeta^j, \quad j\in\{P, E\} \tag{39}$$

We assume that each element in the transfer functions vector is of the form

$$H_i^j(s) = \frac{\prod_{k=1}^{\ell_i^j}(1 + s\omega_{i,k}^j)^{q_{i,k}^j}}{\prod_{k=1}^{m_i^j}(1 + s\tau_{i,k}^j)^{p_{i,k}^j}}, \quad j\in\{P, E\}, \quad i\in N_c^j \tag{40}$$

where ℓ_i^j is the number of distinct zeros and m_i^j is the number of distinct poles in the ith control input transfer function of j. $q_{i,k}^j \geq 1$ and $p_{i,k}^j \geq 1$ are the multiplicity of a zero $-1/\omega_{i,k}^j$ and a pole $-1/\tau_{i,k}^j$, respectively.

From this point forward we will assume that $\forall j\in\{P, E\}$, $i\in N_c^j$:

1) The poles and zeros of $H_i^j(s)$ ($\{-1/\tau_{i,k}^j\}_{1\leq k\leq m_i^j}$ and $\{-1/\omega_{i,k}^j\}_{1\leq k\leq \ell_i^j}$, respectively) may be real or appear in complex conjugate pairs.

2) $H_i^j(s)$ is proper (the order of the denominator polynomial is greater than or equal to the order of the numerator polynomial)

$$\sum_{k=1}^{\ell_i^j} q_{i,k}^j \leq \sum_{k=1}^{m_i^j} p_{i,k}^j$$

3) The poles of $H_i^j(s)$ ($\{-1/\tau_{i,k}^j\}_{1\leq k\leq m_i^j}$) lie on the open left half of the complex plane, yielding

$$\Re\{\tau_{i,1}^j\}, \ldots, \Re\{\tau_{i,m_i^j}^j\} > 0$$

Remark IV.2: From the complex conjugate root theorem, the first assumption is equivalent to the supposition that the numerator and denominator polynomials contain only real coefficients.

Under these assumptions, using partial fraction decomposition, the control dynamics can always be written as

$$H_i^j(s) = d_i^j + \sum_{k=1}^{m_i^j}\sum_{\ell=1}^{p_{i,k}^j} \frac{\lambda_{i,k,\ell}^j}{(1+s\tau_{i,k}^j)^\ell}, \quad j\in\{P,E\}, \quad i\in N_c^j \tag{41}$$

where d_i^j represents the normalized direct lift on j produced by the ith control. For each j and i the constants d_i^j, $\{\lambda_{i,k,\ell}^j\}_{1\leq\ell\leq p_{i,k}^j, 1\leq k\leq m_i^j}$ are the solution of a $(\sum_{k=1}^{m_i^j} p_{i,k}^j + 1)\times(\sum_{k=1}^{m_i^j} p_{i,k}^j + 1)$ system of linear equations, and are functions of the control dynamics constants $\tau_{i,1}^j, \ldots, \tau_{i,m_i^j}^j$ and $\omega_{i,1}^j, \ldots, \omega_{i,\ell_i^j}^j$. Rewriting Eq. (41) with a common denominator and comparing it with Eq. (40) by equating coefficients, we deduce the following:

1) $d_i^j \in \mathbb{R}$ (easily observed, based on Remark IV.2).
2) $d_i^j \neq 0$ iff $H_i^j(s)$ is biproper ($\ell_i^j = m_i^j$).
3) $d_i^j = 0$ iff $H_i^j(s)$ is strictly proper ($\ell_i^j < m_i^j$).
4) $d_i^j = 1$ iff $H_i^j(s) \equiv 1$ (the dynamics of js ith control are ideal).

Recalling Eqs. (16), (20), and (21) we find, through use of the Laplace Transform, that $\forall j\in\{P,E\}$

$$\begin{aligned} \boldsymbol{D}_j\boldsymbol{\Phi}_j(t_f,t)\boldsymbol{B}_j &\equiv \boldsymbol{D}_j\boldsymbol{\Phi}_j(t_f-t)\boldsymbol{B}_j \\ &= \mathcal{L}^{-1}\left\{\frac{\boldsymbol{c}_\zeta^j(s\boldsymbol{I}-\boldsymbol{A}_\zeta^j)^{-1}\boldsymbol{B}_\zeta^j\cos(\gamma_j^{\mathrm{col}}) + \boldsymbol{d}_\zeta^j\cos(\gamma_j^{\mathrm{col}})}{s^2}\right\} \\ &\equiv \mathcal{L}^{-1}\left\{\frac{\boldsymbol{H}_j(s)}{s^2}\right\}\cdot\cos(\gamma_j^{\mathrm{col}}) \end{aligned} \tag{42}$$

and

$$\mathcal{L}^{-1}\{H_i^j(s)\} = d_i^j\delta(\theta) + \sum_{k=1}^{m_i^j}\sum_{\ell=1}^{p_{i,k}^j}\lambda_{i,k,\ell}^j\cdot\frac{\theta^{\ell-1}}{(\tau_{i,k}^j)^p(\ell-1)!}e^{-\theta/\tau_{i,k}^j}, \quad j\in\{P,E\}, \quad i\in N_c^j \tag{43}$$

where $\delta(\theta)$ is a unit impulse at $\theta=0$. $\mathcal{L}^{-1}\{H_i^j(s)/s^2\}$ is obtained from $\mathcal{L}^{-1}\{H_i^j(s)\}$ after replacing the exponent with its Maclaurin series and double integration:

$$\mathcal{L}^{-1}\left\{\frac{H_i^j(s)}{s^2}\right\} = d_i^j\theta + \sum_{k=1}^{m_i^j}\sum_{\ell=1}^{p_{i,k}^j}\lambda_{i,k,\ell}^j\cdot\frac{\tau_{i,k}^j}{(\ell-1)!}\cdot\psi\Big(\theta,\tau_{i,k}^j,\ell\Big), \quad j\in\{P,E\} \tag{44}$$

where

$$\psi(\theta,\tau,\ell) = \sum_{k=0}^{\infty}(-1)^k\cdot\frac{(\theta/\tau)^{\ell+k+1}}{k!(\ell+k)(\ell+k+1)} \tag{45}$$

Since $f_i^j(\theta) \equiv \mathcal{L}^{-1}\{H_i^j(s)/s^2\}\cdot\cos(\gamma_j^{\mathrm{col}})$ is in a fact a function of the time-to-go, we obtain

$$F_j(t_{\mathrm{go}}) = \sum_{i=1}^{n_c^j}\gamma_i^j\cdot|f_i^j(t_{\mathrm{go}})|, \qquad j\in\{P,E\} \tag{46}$$

and Eq. (38) becomes

$$\lim_{t_{\mathrm{go}}\to 0^+}[u_P^{\max}F_P(t_{\mathrm{go}}) - u_E^{\max}F_E(t_{\mathrm{go}})] \geq 0 \tag{47}$$

Remark IV.3: It is important to note that $\forall i\in N_c^j$ even though $f_i^j(\theta)$ may contain complex parameters, since $\boldsymbol{A}_\zeta^j$, $\boldsymbol{B}_\zeta^j$, $\boldsymbol{c}_\zeta^j$ and $\boldsymbol{d}_\zeta^j$ are real, $f_i^j(t_{\mathrm{go}}), F_j(t_{\mathrm{go}})\in\mathbb{R}$ $\forall t_{\mathrm{go}}$.

We observe that, since $f_i^j(t_{\mathrm{go}})$ is obviously smooth and by definition $f_i^j(0)\equiv 0$, the first term in the existing Maclaurin series for each $f_i^j(t_{\mathrm{go}})$ may be omitted, yielding

$$F_j(t_{\mathrm{go}}) = \sum_{i=1}^{n_c^j}\gamma_i^j\cdot\left|\sum_{\alpha=1}^{\infty}\left[\frac{d^\alpha f_i^j(\theta)}{d\theta^\alpha}\right]_{\theta=0}\cdot\frac{t_{\mathrm{go}}^\alpha}{\alpha!}\right|, \quad j\in\{P,E\} \tag{48}$$

and that $\forall j\in\{P,E\}$, $i\in N_c^j$

$$\left.\frac{\mathrm{d}f_i^j(\theta)}{\mathrm{d}\theta}\right|_{\theta=0} = d_i^j\cdot\cos(\gamma_j^{\mathrm{col}}) \tag{49a}$$

$$\left.\frac{\mathrm{d}^\alpha f_i^j(\theta)}{\mathrm{d}\theta^\alpha}\right|_{\theta=0} = \sum_{k=1}^{m_i^j}\sum_{\ell=1}^{p_{i,k}^j}(-1)^{\alpha-\ell-1}\cdot\mathbb{1}(\alpha,\ell)\cdot\frac{\lambda_{i,k,\ell}^j}{(\tau_{i,k}^j)^{\alpha-1}}\cdot\cos(\gamma_j^{\mathrm{col}}), \quad \alpha\geq 2 \tag{49b}$$

where $\mathbb{1}(\alpha,\ell)$ is the indicator function defined by

$$\mathbb{1}(\alpha,\ell) = \begin{cases} 1, & \alpha\geq\ell+1 \\ 0, & \alpha<\ell+1 \end{cases} \tag{50}$$

We define the following additional parameters $\forall j\in\{P,E\}$

$$\delta_j = \sum_{i=1}^{n_c^j}\gamma_i^j\cdot|d_i^j| \tag{51a}$$

$$\sigma_\alpha^j = \sum_{i=1}^{n_c^j}\gamma_i^j\cdot\left|\frac{\mathrm{d}^{\alpha+1}f_i^j(\theta)}{\mathrm{d}\theta^{\alpha+1}}\right|_{\theta=0}, \qquad \alpha\geq 1 \tag{51b}$$

$$\rho_{\alpha,\beta}^j = \sum_{i=1}^{n_c^j}\gamma_i^j\cdot\mathrm{sign}\left\{\left[\frac{\mathrm{d}^{\beta+1}f_i^j(\theta)}{\mathrm{d}\theta^{\beta+1}}\right]_{\theta=0}\right\}\cdot\left[\frac{\mathrm{d}^{\alpha+1}f_i^j(\theta)}{\mathrm{d}\theta^{\alpha+1}}\right]_{\theta=0}, \quad \alpha\geq 1, \quad \beta\geq 0 \tag{51c}$$

and nondimensional parameters

$$\mu = \frac{u_P^{\max}\cdot|\cos(\gamma_P^{\mathrm{col}})|}{u_E^{\max}\cdot|\cos(\gamma_E^{\mathrm{col}})|} \tag{52a}$$

$$\eta = \frac{\delta_P}{\delta_E} \tag{52b}$$

$$\epsilon_\alpha = \frac{\sigma_\alpha^P}{\sigma_\alpha^E} \tag{52c}$$

$$\nu_{\alpha,\beta} = \frac{\rho_{\alpha,\beta}^P}{\rho_{\alpha,\beta}^E} \tag{52d}$$

which, following Remark IV.3, are all real. In effect δ_j represents the maximal possible direct lift on j and η is the pursuer/evader maximal possible direct lift ratio. σ^j_α and $\rho^j_{\alpha,\beta}$ are generally quite complex functions of j's transfer functions' zeros and poles. In the simple case of first-order strictly proper control dynamics, which include a single negative pole $-1/\tau_j$, the sizes of σ^j_1 and $\rho^j_{1,\beta}$ are equal to that of the pole (see Sec. V.B). μ represents the ratio between the pursuer's and evader's maneuverability vertical to the fixed reference line.

Lemma IV.2. If δ_P, $\delta_E \neq 0$ then the necessary condition for the existence of a capture zone is

$$\mu\eta \geq 1$$

and a sufficient condition is

$$\mu\eta > 1$$

Proof: If δ_P, $\delta_E \neq 0$ then for a small-enough t_{go}

$$F_j(t_{\mathrm{go}}) \sim \sum_{i=1}^{n^j_c} \gamma^j_i \cdot \left|\frac{\mathrm{d}f^j_i(\theta)}{\mathrm{d}\theta}\right|_{\theta=0} \cdot t_{\mathrm{go}}, \qquad j \in \{P, E\}$$

Therefore, the necessary condition to satisfy Eq. (47) is

$$u^{\max}_P \sum_{i=1}^{n^P_c} \gamma^P_i \cdot \left|\frac{\mathrm{d}f^P_i(\theta)}{\mathrm{d}\theta}\right|_{\theta=0} \geq u^{\max}_E \sum_{i=1}^{n^E_c} \gamma^E_i \cdot \left|\frac{\mathrm{d}f^E_i(\theta)}{\mathrm{d}\theta}\right|_{\theta=0}$$

and a sufficient condition is

$$u^{\max}_E \sum_{i=1}^{n^P_c} \gamma^P_i \cdot \left|\frac{\mathrm{d}f^P_i(\theta)}{\mathrm{d}\theta}\right|_{\theta=0} > u^{\max}_E \sum_{i=1}^{n^E_c} \gamma^E_i \cdot \left|\frac{\mathrm{d}f^E_i(\theta)}{\mathrm{d}\theta}\right|_{\theta=0}$$

Substituting Eqs. (49a) and (51a) the necessary and sufficient conditions become, respectively,

$$u^{\max}_P \delta_P \cdot |\cos(\gamma^{\mathrm{col}}_P)| \geq u^{\max}_E \delta_E \cdot |\cos(\gamma^{\mathrm{col}}_E)|$$

and

$$u^{\max}_P \delta_P \cdot |\cos(\gamma^{\mathrm{col}}_P)| > u^{\max}_E \delta_E \cdot |\cos(\gamma^{\mathrm{col}}_E)|$$

Dividing both sides by the (positive) right-hand-side expression and substituting Eqs. (52a) and (52b) concludes the proof. ☐

Remark IV.4: Note that

1) If $\delta_E = 0$ ($\{H^E_i(s)\}_{i\in N^E_c}$ are all strictly proper), then the sufficient condition becomes simply $\delta_P \neq 0$. From the definition of δ_P we may further simplify this to $\exists i \in N^P_c{:}d^P_i \neq 0$; that is, there exists a pursuer control that has a nonstrictly proper transfer function.

2) Fixing, for example, the evader's speed and initial heading, the pursuer will benefit from increasing its own speed, thereby increasing the size of $|\cos(\gamma^{\mathrm{col}}_P)|$ and, as a result, increasing its maneuverability relative to the evader's vertical to the reference line. Hence, a slower pursuer will require a greater maneuver capability.

3) In terms of increasing their maneuverability vertical to the reference line, both adversaries benefit from imposing head-on, tail-chase, or head pursuit (in which case $|\cos(\gamma^{\mathrm{col}}_j)| \to 1\ \forall j \in \{P, E\}$).

Lemma IV.3: If $\delta_P = \delta_E = 0$, $\sigma^P_\alpha = \sigma^E_\alpha = 0\ \forall 1 \leq \alpha \leq c-1$, and σ^P_c, $\sigma^E_c \neq 0$ for some $c \geq 1$, then the necessary condition for the existence of a capture zone is

$$\mu\epsilon_c \geq 1$$

and a sufficient condition is

$$\mu\epsilon_c > 1$$

Proof: Continuing the logic of Lemma IV.2, since in this case $\mathrm{d}^\alpha f^j_i(\theta)/\mathrm{d}t^\alpha_{\mathrm{go}}|_{\theta=0} = 0\ \forall 1 \leq \alpha \leq c\ \forall j \in \{P, E\},\ i \in N^j_c$ for some $c \geq 1$, then for a small-enough t_{go}

$$F_j(t_{\mathrm{go}}) \sim \sum_{i=1}^{n^j_c} \gamma^j_i \cdot \left|\frac{\mathrm{d}^{c+1}f^j_i(\theta)}{\mathrm{d}\theta^{c+1}}\right|_{\theta=0} \cdot \frac{t^{c+1}_{\mathrm{go}}}{(c+1)!}$$

Therefore, the necessary condition to satisfy Eq. (47) is

$$u^{\max}_P \textstyle\sum_{i=1}^{n^P_c} \gamma^P_i \cdot \left|\frac{\mathrm{d}^{c+1}f^P_i(\theta)}{\mathrm{d}\theta^{c+1}}\right|_{\theta=0} \geq u^{\max}_E \sum_{i=1}^{n^E_c} \gamma^E_i \cdot \left|\frac{\mathrm{d}^{c+1}f^E_i(\theta)}{\mathrm{d}\theta^{c+1}}\right|_{\theta=0}$$

and a sufficient condition is

$$u^{\max}_P \textstyle\sum_{i=1}^{n^P_c} \gamma^P_i \cdot \left|\frac{\mathrm{d}^{c+1}f_P(\theta)}{\mathrm{d}\theta^{c+1}}\right|_{\theta=0} > u^{\max}_E \sum_{i=1}^{n^E_c} \gamma^E_i \cdot \left|\frac{\mathrm{d}^{c+1}f_E(\theta)}{\mathrm{d}\theta^{c+1}}\right|_{\theta=0}$$

Substituting Eq. (52b) the necessary and sufficient conditions become, respectively,

$$u^{\max}_P \sigma^P_c \cdot |\cos(\gamma^{\mathrm{col}}_P)| \geq u^{\max}_E \sigma^E_c \cdot |\cos(\gamma^{\mathrm{col}}_E)|$$

and

$$u^{\max}_P \sigma^P_c \cdot |\cos(\gamma^{\mathrm{col}}_P)| > u^{\max}_E \sigma^E_c \cdot |\cos(\gamma^{\mathrm{col}}_E)|$$

Dividing both sides by the (positive) right-hand-side expression and substituting Eqs. (52a) and (52c) concludes the proof. ☐

Remark IV.5: Note that

1) Typically, unless j has ideal dynamics, $\sigma^j_\alpha = 0\ \forall 1 \leq \alpha \leq c-1$ for $c > 1$ will not occur, but it may be the case in some special cases.

2) Similar to Remark IV.4, if $\sigma^E_c = 0$ then the sufficient condition becomes simply $\sigma^P_c \neq 0$.

Lemma IV.4: If $\delta_P, \delta_E \neq 0, \mu\eta = 1$, and $\mu\rho^P_{\alpha,0} = \rho^E_{\alpha,0}$ (or $\mu\nu_{\alpha,0} = 1$ if $\rho^P_{\alpha,0}, \rho^E_{\alpha,0} \neq 0$) $\forall 1 \leq \alpha \leq c-1$ for some $c \geq 1$, then the necessary condition for the existence of a capture zone is

$$\mathrm{sign}\{\rho^P_{c,0}\} \cdot \mu|\nu_{c,0}| \geq \mathrm{sign}\{\rho^E_{c,0}\}$$

and a sufficient condition is

$$\mathrm{sign}\{\rho^P_{c,0}\} \cdot \mu|\nu_{c,0}| > \mathrm{sign}\{\rho^E_{c,0}\}$$

Proof: Since $[\mathrm{d}^\alpha f^j_i(\theta)/\mathrm{d}\theta^\alpha]_{\theta=0}$ is finite $\forall \alpha \geq 1$, then for any $c \geq 1$ there is an appropriately small t_{go} such that $\forall j \in \{P, E\}$, $i \in N^j_c$

$$F_j(t_{\mathrm{go}}) \sim \sum_{i=1}^{n^j_c} \gamma^j_i \cdot \left|\sum_{\alpha=1}^{c+1}\left[\frac{\mathrm{d}^\alpha f^j_i(\theta)}{\mathrm{d}\theta^\alpha}\right]_{\theta=0} \cdot \frac{t^\alpha_{\mathrm{go}}}{\alpha!}\right|$$

and

$$\mathrm{sign}\left\{\sum_{\alpha=1}^{c+1}\left[\frac{\mathrm{d}^\alpha f^j_i(\theta)}{\mathrm{d}\theta^\alpha}\right]_{\theta=0} \cdot \frac{t^\alpha_{\mathrm{go}}}{\alpha!}\right\} = \mathrm{sign}\left\{\left[\frac{\mathrm{d}f^j_i(\theta)}{\mathrm{d}\theta}\right]_{\theta=0}\right\}$$

Therefore, after substituting Eqs. (49a) and (51c) and noting that, by definition, $|(\cdot)| = \mathrm{sign}\{(\cdot)\} \cdot (\cdot)$,

$$F_j(t_{\mathrm{go}}) \sim \left[\delta_j \cdot t_{\mathrm{go}} + \sum_{\alpha=2}^{c+1} \rho^j_{\alpha-1,0} \cdot \frac{t^\alpha_{\mathrm{go}}}{\alpha!}\right] \cdot |\cos(\gamma^{\mathrm{col}}_j)|, \quad j \in \{P, E\}$$

If $\mu\eta = 1$ and $\mu\rho^P_{\alpha,0} = \rho^E_{\alpha,0}\ \forall 1 \leq \alpha \leq c-1$, then

$$\begin{aligned} u^{\max}_P&\left[\delta_P \cdot t_{\mathrm{go}} + \sum_{\alpha=2}^{c} \rho^P_{\alpha-1,0} \cdot \frac{t^\alpha_{\mathrm{go}}}{\alpha!}\right] \cdot |\cos(\gamma^{\mathrm{col}}_P)| \\ &= u^{\max}_E\left[\delta_E \cdot t_{\mathrm{go}} + \sum_{\alpha=2}^{c} \rho^E_{\alpha-1,0} \cdot \frac{t^\alpha_{\mathrm{go}}}{\alpha!}\right] \cdot |\cos(\gamma^{\mathrm{col}}_E)| \triangleq \Delta_c(t_{\mathrm{go}}) \end{aligned}$$

and

$$u_P^{\max} F_P(t_{go}) \sim \Delta_c(t_{go}) + u_P^{\max} \rho_{c,0}^P \cdot \frac{t_{go}^{c+1}}{(c+1)!} \cdot |\cos(\gamma_P^{col})|$$

$$u_E^{\max} F_E(t_{go}) \sim \Delta_c(t_{go}) + u_E^{\max} \rho_{c,0}^E \cdot \frac{t_{go}^{c+1}}{(c+1)!} \cdot |\cos(\gamma_E^{col})|$$

Therefore, the necessary condition to satisfy Eq. (47) is

$$u_P^{\max} \rho_{c,0}^P \cdot |\cos(\gamma_P^{col})| \geq u_E^{\max} \rho_{c,0}^E \cdot |\cos(\gamma_E^{col})|$$

and a sufficient condition is

$$u_P^{\max} \rho_{c,0}^P \cdot |\cos(\gamma_P^{col})| > u_E^{\max} \rho_{c,0}^E \cdot |\cos(\gamma_E^{col})|$$

Dividing both sides by $u_E^{\max} |\rho_{c,0}^E| \cdot |\cos(\gamma_E^{col})|$ and substituting Eqs. (52a) and (52d) concludes the proof. □

Remark IV.6: If in a case, such as the one presented in Lemma IV.4, $\text{sign}\{\rho_{c,0}^P\} \neq \text{sign}\{\rho_{c,0}^E\}$, then the necessary and sufficient conditions for the existence of a capture zone are

a) nonexistent if $\text{sign}\{\rho_{c,0}^P\} < \text{sign}\{\rho_{c,0}^E\}$

b) any $\mu > 0$, $|\nu_{c,0}| \geq 0$ if $\text{sign}\{\rho_{c,0}^P\} > \text{sign}\{\rho_{c,0}^E\}$.

Lemma IV.5: If $\delta_P = \delta_E = 0$, $\sigma_\alpha^P = \sigma_\alpha^E = 0\ \forall 1 \leq \alpha \leq r-1$, σ_r^P, $\sigma_r^E \neq 0$, and $\mu \rho_{\beta,r}^P = \rho_{\beta,r}^E$ (or $\mu \nu_{\beta,r} = 1$ if $\rho_{\beta,r}^P, \rho_{\beta,r}^E \neq 0$) $\forall r+1 \leq \beta \leq c-1$ for some $c \geq r \geq 1$, then the necessary condition for the existence of a capture zone is

$$\text{sign}\{\rho_{c,r}^P\} \cdot \mu |\nu_{c,r}| \geq \text{sign}\{\rho_{c,r}^E\}$$

and a sufficient condition is

$$\text{sign}\{\rho_{c,r}^P\} \cdot \mu |\nu_{c,r}| > \text{sign}\{\rho_{c,r}^E\}$$

Proof: The proof follows the logic of the proofs for Lemma IV.3 and Lemma IV.4, treating σ_r^j as δ_j. □

Remark IV.7: If in a case, such as the one presented in Lemma IV.5, $\text{sign}\{\rho_{c,r}^P\} \neq \text{sign}\{\rho_{c,r}^E\}$, then the necessary and sufficient conditions for the existence of a capture zone are

a) nonexistent if $\text{sign}\{\rho_{c,r}^P\} < \text{sign}\{\rho_{c,r}^E\}$,

b) any $\mu > 0$, $|\nu_{c,r}| \geq 0$ if $\text{sign}\{\rho_{c,r}^P\} > \text{sign}\{\rho_{c,r}^E\}$.

V. Some Examples

We now present some results for several cases in order to examine and verify the validity of the theory developed in the preceding section. As previously stated, the optimality of solutions to pursuit-evasion games suggests that, in the same framework, the capturability conditions must coincide with conditions for the existence of a capture zone in the game. For this reason we chose the following previously studied examples, each with known linear-pursuit-evasion-games-based results.

The following examples cover different control types, including variations on the number of control inputs and the modeling of the direct lift and airframe response as a result of command inputs. In the present work we will limit ourselves to a maximum of two control inputs, each of which may be categorized as either forward/canard control or tail control. A general case including both types of control inputs is depicted in Fig. 2. We assume that each adversary's center of pressure (CP), where the total force F_b is obtained, is located behind its center of mass (CG), and as a result the airframe itself is statically stable. As initially stated, any aerodynamic coupling between the lateral forces (along Z_b) produced by the control surfaces is neglected and it is assumed that the adversaries do not accelerate (the total force along X_b is zero).

A. Ideal Control Dynamics

As in [7], assume that each adversary has a single control input and ideal control dynamics. This would represent adversaries for which the control inputs are immediately translated into direct lift with no airframe response; that is, the control action takes place at the center of gravity. In such a case $n_c^P = n_c^E = 1$, $\ell_j = m_j = 0\ \forall j \in \{P, E\}$,

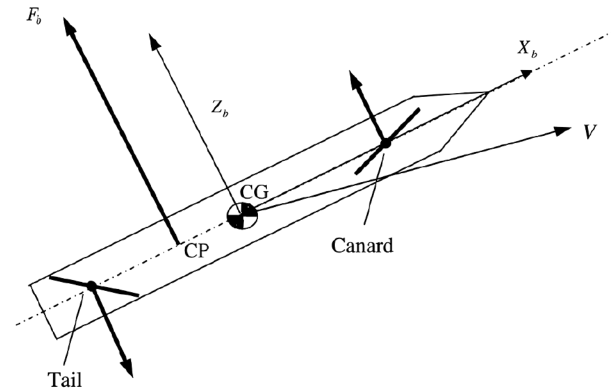

Fig. 2 Tail- and canard-controlled missile schematic.

$$H_j(s) = 1, \qquad j \in \{P, E\}$$

and from Eq. (44)

$$f_j(\theta) = d_j \theta \cdot \cos(\gamma_j^{col}), \qquad j \in \{P, E\}$$

As previously deduced, in such a case $d_j = 1\ \forall j \in \{P, E\}$; therefore,

$$f_j(\theta) = \theta \cdot \cos(\gamma_j^{col}), \qquad j \in \{P, E\}$$

for which

$$\begin{matrix} \delta_j = 1 \\ \sigma_\alpha^j = 0 \end{matrix}, \qquad \alpha \geq 1, \qquad j \in \{P, E\}$$

Substituting into Eq. (52b) we have

$$\eta = 1$$

Following Lemma IV.2 the necessary and sufficient condition for the existence of a capture zone is simply $\mu \geq 1$, meaning that relative to the reference line the pursuer is required to have a maneuverability advantage over the evader, in accordance with the results presented in [7].

B. First-Order Strictly Proper Control Dynamics

Now assume, as in [9], that each adversary has a single control input, the dynamics of which include a single real and positive time constant. This would be a possible representation of two adversaries where there exists negligible direct lift from the control surface (tail or canard); that is, it mainly produces a moment for rotating the airframe (by being located far from the center of gravity). In this case neither has direct lift, but their airframe responses are modeled. In this instance $n_c^P = n_c^E = 1$, $\ell_j = 0$, $m_j = 1$, $p_j = 1\ \forall j \in \{P, E\}$,

$$H_j(s) = \frac{1}{1 + s\tau_j}, \qquad j \in \{P, E\}$$

and from Eq. (44)

$$f_j(\theta) = \left[d_j \theta + \tau_j \psi(\theta, \tau_j, 1) \right] \cdot \cos(\gamma_j^{col}), \qquad j \in \{P, E\}$$

In this case, as deduced for proper dynamics, $d_j = 0\ \forall j \in \{P, E\}$ and from Eq. (45)

$$\psi(\theta,\tau_j,1)=\sum_{k=0}^{\infty}(-1)^k\cdot\frac{(\theta/\tau_j)^{k+2}}{k!(k+1)(k+2)}$$
$$=\sum_{m=0}^{\infty}(-1)^m\cdot\frac{(\theta/\tau_j)^m}{m!}-(-1)^1\cdot\frac{\theta}{\tau_j}-(-1)^0\cdot 1$$
$$=e^{-\theta/\tau_j}+\frac{\theta}{\tau_j}-1 \tag{53}$$

Substituting into the expression for $f_j(\theta)$ yields

$$f_j(\theta)=\tau_j\left(e^{-\theta/\tau_j}+\frac{\theta}{\tau_j}-1\right)\cdot\cos(\gamma_j^{\mathrm{col}}),\qquad j\in\{P,E\}$$

for which

$$\delta_j=0,\quad \sigma_\alpha^j=\frac{1}{(\tau_j)^\alpha},\quad \rho_{\alpha,1}^j=(-1)^{\alpha-1}\cdot\frac{1}{(\tau_j)^\alpha},\qquad \alpha\geq 1,\qquad j\in\{P,E\}$$

Substituting into Eqs. (52c) and (52d) we have

$$\epsilon_\alpha=\nu_{\alpha,1}=\left(\frac{\tau_E}{\tau_P}\right)^\alpha\triangleq\varepsilon^\alpha,\qquad \alpha\geq 1$$

where ε is the evader/pursuer time constant ratio. Since $\delta_P=\delta_E=0$, the sufficient condition for the existence of a capture zone, according to Lemma IV.3 and Lemma IV.5, is in general

a) $\mu\epsilon_1\geq 1$

b) $\mathrm{sign}\{\rho_{c,1}^P\}\cdot\mu|\nu_{c,1}|\geq\mathrm{sign}\{\rho_{c,1}^E\}1$ if $\mu\nu_{\alpha,1}=1\ \forall 1\leq\alpha\leq c-1\ \forall c\geq 2$,

which in this example becomes

a) $\mu\varepsilon\geq 1$ (denoted by some as an agility advantage)

b) $(-1)^{c-1}\cdot\mu\varepsilon^c\geq(-1)^{c-1}$ if $\mu\varepsilon^\alpha=1\ \forall 1\leq\alpha\leq c-1$.

Note that $\forall c\geq 3$ if $\mu\varepsilon^\alpha=1\ \forall 1\leq\alpha\leq c-1$ then $\varepsilon=1$. Therefore, from cases $c=1$ and $c=2$, after substituting $\mu\varepsilon=1$, we can deduce the following sufficient condition:

a) $\mu\varepsilon>1$

b) $\varepsilon\leq 1$ if $\mu\varepsilon=1$.

In [10] it was shown that in the framework of this example the sufficient conditions for the existence of a capture zone are

a) $\mu\varepsilon>1$

b) $\mu\geq 1$ if $\mu\varepsilon=1$.

In the second case note that since $\varepsilon=1/\mu$, then $\mu\geq 1$ and $\varepsilon\leq 1$ are equivalent.

Furthermore, in accordance with Lemma IV.3, it was shown in [8] that if $\tau_E=0$ (ideal evader) then a capture zone cannot exist (since $0=\mu\varepsilon<1$).

C. First-Order Biproper Control Dynamics

In this case, as in [14], assume that each adversary has a single control with biproper dynamics, defined by a single real zero and a single negative pole. This model essentially generalizes the two previous ones. It could be used to represent adversaries with either tail or forward/canard control having both direct lift and airframe response elements. In this case $n_c^P=n_c^E=1$, $\ell_j=m_j=1$, $p_j=q_j=1\ \forall j\in\{P,E\}$,

$$H_j(s)=\frac{1+s\omega_j}{1+s\tau_j},\qquad j\in\{P,E\}$$

and from Eqs. (44) and (53)

$$f_j(\theta)=\left[\frac{\omega_j}{\tau_j}\theta+\tau_j\left(1-\frac{\omega_j}{\tau_j}\right)\psi(\theta,\tau_j,1)\right]\cdot\cos(\gamma_j^{\mathrm{col}})$$
$$=\left[\frac{\omega_j}{\tau_j}\theta+\tau_j\left(1-\frac{\omega_j}{\tau_j}\right)\cdot\left(e^{-\theta/\tau_j}+\frac{\theta}{\tau_j}-1\right)\right]\cdot\cos(\gamma_j^{\mathrm{col}}),\quad j\in\{P,E\}$$

for which

$$\delta_j=\frac{\omega_j}{\tau_j}\triangleq d_j,\quad \sigma_\alpha^j=\frac{|1-d_j|}{(\tau_j)^\alpha},\quad \rho_{\alpha,0}^j=(-1)^{\alpha-1}\cdot\mathrm{sign}\{d_j\}\cdot\frac{1-d_j}{(\tau_j)^\alpha},\qquad \alpha\geq 1,\qquad j\in\{P,E\}$$

Substituting into Eqs. (52b), (52c), and (52d) we obtain

$$\eta=\left|\frac{d_P}{d_E}\right|,\quad \epsilon_\alpha=\left|\frac{1-d_P}{1-d_E}\right|\cdot\left(\frac{\tau_E}{\tau_P}\right)^\alpha,\quad \nu_{\alpha,0}=\mathrm{sign}\left\{\frac{d_P}{d_E}\right\}\cdot\frac{1-d_P}{1-d_E}\cdot\left(\frac{\tau_E}{\tau_P}\right)^\alpha,\qquad \alpha\geq 1$$

Since $\delta_P,\delta_E\neq 0$, then, according to Lemma IV.2 and Lemma IV.4, the sufficient condition for the existence of a capture zone is in general

a) $\mu\eta>1$

b) $\mathrm{sign}\{\rho_{c,0}^P\}\cdot\mu|\nu_{c,0}|\geq\mathrm{sign}\{\rho_{c,0}^E\}$ if $\mu\eta=1$, $\mu\nu_{\alpha,0}=1\ \forall 1\leq\alpha\leq c-1\ \forall c\geq 1$,

which in the examined case becomes

a) $\mu\eta>1$

b) $(-1)^{c-1}\cdot\mathrm{sign}\{d_P\}\cdot\mathrm{sign}\{1-d_P\}\cdot\mu\varepsilon^c|(1-d_P)/(1-d_E)|\geq(-1)^{c-1}\cdot\mathrm{sign}\{d_E\}\cdot\mathrm{sign}\{1-d_E\}$ if $\mu\eta=1$, $\mu\varepsilon^\alpha|(1-d_P)/(1-d_E)|=1\ \forall 1\leq\alpha\leq c-1\ \forall c\geq 1$ and $\mathrm{sign}\{d_P/d_E\}=\mathrm{sign}\{(1-d_P)/(1-d_E)\}$

c) nonexistent if $\mu\eta=1$ and $\mathrm{sign}\{d_P\}\cdot(1-d_P)<0<\mathrm{sign}\{d_E\}\cdot(1-d_E)$

d) any μ, $\varepsilon>0$ if $\mu\eta=1$ and $\mathrm{sign}\{d_P\}\cdot(1-d_P)>0>\mathrm{sign}\{d_E\}\cdot(1-d_E)$,

where $\varepsilon=\tau_E/\tau_P$ is, once again, the evader/pursuer time constant ratio. Similar to the second example, note that $\forall c\geq 3$ if $\mu\eta=1$, $\mu\varepsilon^\alpha|(1-d_P)/(1-d_E)|=1\ \forall 1\leq\alpha\leq c-1$ and $\mathrm{sign}\{d_P/d_E\}=\mathrm{sign}\{(1-\delta_P)/(1-\delta_E)\}$ then $\varepsilon=1$. Therefore, from cases $c=1$ and $c=2$, after substituting $\mu\varepsilon|(1-d_P)/(1-d_E)|=1$, we can deduce the following sufficient condition:

a) $\mu\eta>1$

b) $\mathrm{sign}\{d_P\}\cdot\mathrm{sign}\{1-d_P\}\mu\varepsilon|(1-d_P)/(1-d_E)|>\mathrm{sign}\{d_E\}\cdot\mathrm{sign}\{(1-d_E)\}$ if $\mu\eta=1$ and $\mathrm{sign}\{d_P/d_E\}=\mathrm{sign}\{(1-d_P)/(1-d_E)\}$

c) $\mathrm{sign}\{d_P\}\cdot\mathrm{sign}\{1-d_P\}\varepsilon\leq\mathrm{sign}\{d_E\}\cdot\mathrm{sign}\{1-d_E\}$ if $\mu\eta=1$, $\mu\varepsilon|(1-d_P)/(1-d_E)|=1$ and $\mathrm{sign}\{d_P/d_E\}=\mathrm{sign}\{(1-d_P)/(1-d_E)\}$

d) nonexistent if $\mu\eta=1$ and $\mathrm{sign}\{d_P\}\cdot(1-d_P)<0<\mathrm{sign}\{d_E\}\cdot(1-d_E)$

e) any μ, $\varepsilon>0$ if $\mu\eta=1$ and $\mathrm{sign}\{d_P\}\cdot(1-d_P)>0>\mathrm{sign}\{d_E\}\cdot(1-d_E)$.

As an example let us assume that $-1<d_P,d_E<1$.

Case A: $0<d_P,d_E<1$ (forward/canard-controlled adversaries)

In this case the sufficient condition for the existence of a capture zone is

a) $\mu\eta>1$

b) $\mu\varepsilon|(1-d_P)/(1-d_E)|>1$ if $\mu\eta=1$

c) $\varepsilon\leq 1$ if $\mu\eta=1$ and $\mu\varepsilon|(1-d_P)/(1-d_E)|=1$.

Case B: $-1<d_P,d_E<0$ (tail-controlled adversaries)

In this case the sufficient condition for the existence of a capture zone is

a) $\mu\eta > 1$

b) $\mu\varepsilon|(1-d_P)/(1-d_E)| < 1$ if $\mu\eta = 1$

c) $\varepsilon \geq 1$ if $\mu\eta = 1$ and $\mu\varepsilon|(1-d_P)/(1-d_E)| = 1$.

Case C: $-1 < d_P < 0 < d_E < 1$ (tail-controlled pursuer, forward/canard-controlled evader)

In this case the sufficient condition for the existence of a capture zone is

a) $\mu\eta > 1$

b) nonexistent if $\mu\eta = 1$.

Case D: $-1 < d_E < 0 < d_P < 1$ (forward/canard-controlled pursuer, tail-controlled evader)

In this case the sufficient condition for the existence of a capture zone is

a) $\mu\eta > 1$

b) any $\mu, \varepsilon > 0$ if $\mu\eta = 1$.

where cases A and B present the required advantages of the pursuer against a similarly controlled evader, cases C and D shed some light on scenarios with adversaries with different control types. According to case C a tail-controlled missile cannot hope to capture a forward-controlled evader unless it has a distinct direct lift advantage. However, following case D, a forward/canard-controlled missile could successfully intercept a tail-controlled evader even without a distinct advantage in direct lift. These results point to the superiority, with respect to capturability, of forward control over tail control, supporting existing conclusions [13,14].

These results expand upon those presented in [14], in which it was shown that for adversaries with first-order biproper dynamics, for which $-1 < d_P,\ d_E < 1$, $\mu\eta \geq 1$ is a necessary condition for the existence of a capture zone and that $\mu\eta > 1$ is a sufficient condition.

D. Second-Order Strictly Proper Control Dynamics

We now consider a scenario in which each adversary has a single control input, the dynamics of which include a complex conjugate pair. This once again represents forward/canard-controlled adversaries that have no direct lift, but the airframe response model is more elaborate. In this instance $n_c^P = n_c^E = 1$, $\ell_j = 0$, $m_j = 2$, $p_1^j = p_2^j = 1\ \forall j \in \{P, E\}$,

$$H_j(s) = \frac{1}{(1+s\tau_j)(1+s\tau_j^*)}, \qquad j \in \{P, E\}$$

and Eq. (44)

$$f_j(\theta) = d_j\theta + \lambda_j\tau_j\psi(\theta, \tau_j, 1) + \lambda_j^*\tau_j^*\psi(\theta, \tau_j^*, 1), \qquad j \in \{P, E\}$$

where

$$\begin{aligned} \lambda_j &= -\frac{\tau_j}{\tau_j^* - \tau} \\ \lambda_j^* &= \frac{\tau_j^*}{\tau_j^* - \tau} \end{aligned}, \qquad j \in \{P, E\}$$

and $d_j = 0\ \forall j \in \{P, E\}$, as per the deduction for proper dynamics. $\Re\{\tau_j\} = \Re\{\tau_j^*\} \geq 0\ \forall j \in \{P, E\}$ and, without loss of generality, we assume that $\Im\{\tau_j\} = -\Im\{\tau_j^*\} \geq 0$. After the substitution of Eq. (53) in $f_j(\theta)$ we obtain, after some algebra, $\forall j \in \{P, E\}$

$$\begin{aligned} f_j(\theta) &= e^{-\theta\cdot\Re\{\tau_j\}/(\tau_j\cdot\tau_j^*)} \\ &\cdot\left[2\Re\{\tau_j\}\cdot\cos\left(\frac{\Im\{\tau_j\}}{\tau_j\cdot\tau_j^*}\theta\right) + \frac{\Re\{\tau_j\}^2 - \Im\{\tau_j\}^2}{\Im\{\tau_j\}}\cdot\sin\left(\frac{\Im\{\tau_j\}}{\tau_j\cdot\tau_j^*}\theta\right)\right] \\ &+ \theta - 2\Re\{\tau_j\} \end{aligned}$$

for which

$$\begin{aligned} \delta_j &= \sigma_1^j = 0 \\ \sigma_\alpha^j &= \left|\frac{\tau_j^{\alpha-1} - (\tau_j^*)^{\alpha-1}}{(\tau_j\tau_j^*)^{\alpha-1}(\tau_j^* - \tau_j)}\right| \\ \rho_{\alpha,2}^j &= (-1)^{\alpha-1}\cdot\operatorname{sign}\left\{-\frac{1}{\tau_j\tau_j^*}\right\}\cdot\frac{\tau_j^{\alpha-1} - (\tau_j^*)^{\alpha-1}}{(\tau_j\tau_j^*)^{\alpha-1}(\tau_j^* - \tau_j)} \\ &\alpha \geq 2, \qquad j \in \{P, E\} \end{aligned},$$

Using the known general factorization of the difference of two nth powers [22]

$$a^\alpha - b^\alpha = (a-b)\cdot\sum_{k=0}^{\alpha-1} a^{\alpha-1-k}\cdot b^k$$

we get

$$\begin{aligned} \sigma_\alpha^j &= \left|\frac{\sum_{k=0}^{\alpha-2}\tau_j^{\alpha-2-k}\cdot(\tau_j^*)^k}{(\tau_j\tau_j^*)^{\alpha-1}}\right| \\ \rho_{\alpha,2}^j &= (-1)^{\alpha-1}\cdot\frac{\sum_{k=0}^{\alpha-2}\tau_j^{\alpha-2-k}\cdot(\tau_j^*)^k}{(\tau_j\tau_j^*)^{\alpha-1}} \end{aligned}, \qquad \alpha \geq 2, \quad j \in \{P, E\}$$

Substituting into Eqs. (52c) and (52d) we have

$$\begin{aligned} \epsilon_\alpha &= \left|\frac{\sum_{k=0}^{\alpha-2}\tau_P^{\alpha-2-k}\cdot(\tau_P^*)^k}{\sum_{k=0}^{\alpha-2}\tau_E^{\alpha-2-k}\cdot(\tau_E^*)^k}\right|\cdot\left(\frac{\tau_E\tau_E^*}{\tau_P\tau_P^*}\right)^{\alpha-1} \\ \nu_{\alpha,2} &= \frac{\sum_{k=0}^{\alpha-2}\tau_P^{\alpha-2-k}\cdot(\tau_P^*)^k}{\sum_{k=0}^{\alpha-2}\tau_E^{\alpha-2-k}\cdot(\tau_E^*)^k}\cdot\left(\frac{\tau_E\tau_E^*}{\tau_P\tau_P^*}\right)^{\alpha-1} \end{aligned}, \qquad \alpha \geq 2$$

Since $\delta_P = \delta_E = 0$ and $\sigma_1^P = \sigma_1^E = 0$, the sufficient condition for the existence of a capture zone, according to Lemma IV.3 and Lemma IV.5, is in general $\mu\epsilon_2 \geq 1$ and $\operatorname{sign}\{\rho_{c,2}^P\}\cdot\mu|\nu_{c,2}| \geq \operatorname{sign}\{\rho_{c,2}^E\}$ if $\mu\nu_{\alpha,2} = 1\ \forall 2 \leq \alpha \leq c-1\ \forall c \geq 3$. Unfortunately, a more simplified representation cannot be easily obtained in this example. However, by simply checking the cases $c = 3$ and $c = 4$ we find that the necessary condition can be written as

a) $\mu\tilde{\varepsilon} < 1$

b) $(\Re\{\tau_P\}/\Re\{\tau_E\})\tilde{\varepsilon} > 1$ if $\mu\tilde{\varepsilon} = 1$

c) $(\Im\{\tau_P\}/\Im\{\tau_E\})\tilde{\varepsilon} \geq 1$ if $\mu\tilde{\varepsilon} = 1$ and $(\Re\{\tau_P\}/\Re\{\tau_E\})\tilde{\varepsilon} = 1$,

where

$$\tilde{\varepsilon} = \frac{\tau_E\tau_E^*}{\tau_P\tau_P^*}$$

Note that $\nu_{c,2}$ can be written as

$$\nu_{c,2} = \frac{\sum_{k=0}^{c-2}\tau_P^{c-2-k}\cdot(\tau_P^*)^k\cdot\tilde{\varepsilon}^{c-2}}{\sum_{k=0}^{c-2}\tau_E^{c-2-k}\cdot(\tau_E^*)^k}\cdot\tilde{\varepsilon}$$

or, since $\sum_{k=0}^{c-2}\tau_P^{c-2-k}\cdot(\tau_P^*)^k$ and $\sum_{k=0}^{c-2}\tau_E^{c-2-k}\cdot(\tau_E^*)^k$ are symmetric expressions of $\Re\{\tau_P\}$ and $\Im\{\tau_P\}$ and of $\Re\{\tau_E\}$ and $\Im\{\tau_E\}$, respectively,

$$\nu_{c,2} = \frac{\sum_{k=0}^{c-2}a_k\cdot(\Re\{\tau_P\}\cdot\tilde{\varepsilon})^{c-2-k}\cdot(\Im\{\tau_P\}\cdot\tilde{\varepsilon})^k}{\sum_{k=0}^{c-2}a_k\cdot(\Re\{\tau_E\})^{c-2-k}\cdot(\Im\{\tau_E\})^k}\cdot\tilde{\varepsilon}$$

where a_k is the resulting coefficient of the kth term in the sum. Therefore, $\forall c \geq 5$ if $(\Re\{\tau_P\}/\Re\{\tau_E\})\tilde{\varepsilon} = 1$ and $(\Im\{\tau_P\}/\Im\{\tau_E\})\tilde{\varepsilon} = 1$ we necessarily get $\nu_{c,2} = \tilde{\varepsilon}$, and consequently, if $\mu\tilde{\varepsilon} = 1$ then $\mu|\nu_{c,2}| = 1$. Finally, since $\operatorname{sign}\{\rho_{c,2}^P\} = \operatorname{sign}\{\rho_{c,2}^E\}$ we find that $\operatorname{sign}\{\rho_{c,2}^P\}\cdot\mu|\nu_{c,2}| = \operatorname{sign}\{\rho_{c,2}^E\}\ \forall c \geq 5$ if $\mu|\nu_{c,2}| = 1$, $(\Re\{\tau_P\}/\Re\{\tau_E\})\tilde{\varepsilon} = 1$ and $(\Im\{\tau_P\}/\Im\{\tau_E\})\tilde{\varepsilon} = 1$.

In [16,23] it was shown that in an engagement between a pursuer and an evader with strictly proper second-order and first-order control dynamics, respectively, a capture zone cannot exist. From the presented results for first- and second-order control dynamics we have that in such a case $\delta_P = \delta_E = 0$, $\sigma_1^P = 0$, and $\sigma_1^E \neq 0$. Therefore, according to Lemma IV.3 the necessary capture zone existence condition is not satisfied for $c = 1$, in accordance with the known result. Apart from that, the obtained conditions constitute a substantial extension to the more general case of both adversaries having strictly proper second-order control dynamics.

E. Dual-Controlled Adversaries with First-Order Biproper Control Dynamics

In [15] the case of a dual-controlled pursuer with first-order biproper control dynamics versus a forward/canard-controlled evader with first-order strictly proper control dynamics was examined. Assume in this example a slightly more general case in which each entity has two control inputs, the dynamics of which are described by two biproper transfer functions, each with a single real zero and both with a common single negative pole. This represents two adversaries with both tail and forward/canard control, each of which includes a direct lift component and the airframe response. In this case $n_c^P = n_c^E = 2, \ell_i^j = m_i^j = 1, p_i^j = q_i^j = 1\ \forall j \in \{P, E\}\ \forall i \in \{1, 2\}$,

$$H_i^j(s) = \frac{1 + s\omega_i^j}{1 + s\tau_j}, \qquad j \in \{P, E\}, \qquad i \in \{1, 2\}$$

and from Eqs. (44) and (53) $\forall j \in \{P, E\}$

$$\begin{aligned} f_i^j(\theta) &= d_i^j\theta + \tau_j(1 - d_i^j)\psi(\theta, \tau_j, 1) \\ &= d_i^j\theta + \tau_j(1 - d_i^j) \cdot \left(e^{-\theta/\tau_j} + \frac{\theta}{\tau_j} - 1\right), \quad i \in \{1, 2\} \end{aligned}$$

where

$$d_i^j = \frac{\omega_i^j}{\tau_i^j}, \qquad j \in \{P, E\}, \qquad i \in \{1, 2\}$$

and for which

$$\begin{aligned} \delta_j &= \gamma_1^j \cdot |d_1^j| + \gamma_2^j \cdot |d_2^j| \\ \sigma_\alpha^j &= \frac{\gamma_1^j \cdot |1 - d_1^j| + \gamma_2^j \cdot |1 - d_2^j|}{(\tau_2^j)^\alpha}, \\ \rho_{\alpha,0}^j &= (-1)^{\alpha-1} \cdot \frac{\operatorname{sign}\{d_1^j\} \cdot \gamma_1^j \cdot (1 - d_1^j) + \operatorname{sign}\{d_2^j\} \cdot \gamma_2^j \cdot (1 - d_2^j)}{(\tau_2^j)^\alpha} \\ &\alpha \geq 1, \qquad j \in \{P, E\} \end{aligned}$$

Substituting into Eqs. (52b), (52c), and (52d) yields

$$\begin{aligned} \eta &= \frac{\gamma_1^P \cdot |d_1^P| + \gamma_2^P \cdot |d_2^P|}{\gamma_1^E \cdot |d_1^E| + \gamma_2^E \cdot |d_2^E|} \\ \varepsilon_\alpha &= \frac{\gamma_1^P \cdot |1 - d_1^P| + \gamma_2^P \cdot |1 - d_2^P|}{\gamma_1^E \cdot |1 - d_1^E| + \gamma_2^E \cdot |1 - d_2^E|} \cdot \left(\frac{\tau_E}{\tau_P}\right)^\alpha, \\ \nu_{\alpha,0} &= \frac{\operatorname{sign}\{d_1^P\} \cdot \gamma_1^P \cdot (1 - d_1^P) + \operatorname{sign}\{d_2^P\} \cdot \gamma_2^P \cdot (1 - d_2^P)}{\operatorname{sign}\{d_1^E\} \cdot \gamma_1^E \cdot (1 - d_1^E) + \operatorname{sign}\{d_2^E\} \cdot \gamma_2^E \cdot (1 - d_2^E)} \cdot \left(\frac{\tau_E}{\tau_P}\right)^\alpha \\ &\alpha \geq 1 \end{aligned}$$

In general, according to Lemma IV.2 and Lemma IV.4, since δ_P, $\delta_E \neq 0$, the sufficient condition for the existence of a capture zone is
a) $\mu\eta > 1$
b) $\operatorname{sign}\{\rho_{c,0}^P\} \cdot \mu|\nu_{c,0}| \geq \operatorname{sign}\{\rho_{c,0}^E\}$ if $\mu\eta = 1, \mu\nu_{\alpha,0} = 1\ \forall 1 \leq \alpha \leq c - 1\ \forall c \geq 1$,
which in this example can be written as
a) $\mu\eta > 1$
b) $(-1)^{c-1} \cdot \operatorname{sign}\{\varphi_P\} \cdot \mu\epsilon^c|\varphi_P/\varphi_E| \geq (-1)^{c-1} \cdot \operatorname{sign}\{\varphi_E\}$ if $\mu\eta = 1$, $\mu\varepsilon^\alpha|\varphi_P/\varphi_E| = 1\ \forall 1 \leq \alpha \leq c - 1\ \forall c \geq 1$ and $\operatorname{sign}\{\varphi_P\} = \operatorname{sign}\{\varphi_E\}$
c) nonexistent if $\mu\eta = 1$ and $\varphi_P < 0 < \varphi_E$
d) any $\mu, \varepsilon > 0$ if $\mu\eta = 1$ and $\varphi_P > 0 > \varphi_E$,
where

$$\varphi_j = \operatorname{sign}\{d_1^j\} \cdot \gamma_1^j \cdot (1 - d_1^j) + \operatorname{sign}\{d_2^j\} \cdot \gamma_2^j \cdot (1 - d_2^j), \quad j \in \{P, E\}$$

and recall that $\varepsilon = \tau_E/\tau_P$ is the evader/pursuer time constant ratio. Note that $\forall c \geq 3$ if $\mu\eta = 1$, $\mu\varepsilon^\alpha|\varphi_P/\varphi_E| = 1\ \forall 1 \leq \alpha \leq c - 1$, and $\operatorname{sign}\{\varphi_P\} = \operatorname{sign}\{\varphi_E\}$, then $\varepsilon = 1$. Therefore, from cases $c = 1$ and $c = 2$, after substituting $\mu\varepsilon|\varphi_P/\varphi_E| = 1$, we can deduce the following sufficient condition:
a) $\mu\eta > 1$
b) $\operatorname{sign}\{\varphi_P\}\mu\varepsilon|\varphi_P/\varphi_E| > \operatorname{sign}\{\varphi_E\}$ if $\mu\eta = 1$ and $\operatorname{sign}\{\varphi_P\} = \operatorname{sign}\{\varphi_E\}$
c) $\operatorname{sign}\{\varphi_P\}\varepsilon \leq \operatorname{sign}\{\varphi_E\}$ if $\mu\eta = 1$, $\mu\varepsilon|\varphi_P/\varphi_E| = 1$ and $\operatorname{sign}\{\varphi_P\} = \operatorname{sign}\{\varphi_E\}$
d) nonexistent if $\mu\eta = 1$ and $\varphi_P < 0 < \varphi_E$
e) any $\mu, \varepsilon > 0$ if $\mu\eta = 1$ and $\varphi_P > 0 > \varphi_E$.

As an example let us assume that $0 < d_1^P, d_1^E < 1$ and $-1 < d_2^P, d_2^E < 0$ (such would be the case in an engagement between two dual-controlled missiles, where each couplet d_1^j and d_2^j represented the direct lift of the canard and tail controls of j, respectively [15]). In this case $(1 - d_2^j) > (1 - d_1^j) > 0\ \forall j \in \{P, E\}, i \in N_c^j$ and

$$\varphi_j = \gamma_1^j \cdot (1 - d_1^j) - \gamma_2^j \cdot (1 - d_2^j), \qquad j \in \{P, E\}$$

Case A: $1 < (1 - d_2^j)/(1 - d_1^j) < \gamma_1^j/\gamma_2^j\ \forall j \in \{P, E\}$ (strictly greater canard/forward control influence for both adversaries)

In this case the sufficient condition for the existence of a capture zone is
a) $\mu\eta > 1$
b) $\mu\varepsilon|\varphi_P/\varphi_E| > 1$ if $\mu\eta = 1$
c) $\varepsilon \leq 1$ if $\mu\eta = 1$ and $\mu\varepsilon|\varphi_P/\varphi_E| = 1$.

Case B: $(1 - d_2^j)/(1 - d_1^j) > \gamma_1^j/\gamma_2^j\ \forall j \in \{P, E\}$ (nonstrictly greater canard/forward control influence for both adversaries)

In this case the sufficient condition for the existence of a capture zone is
a) $\mu\eta > 1$
b) $\mu\varepsilon|\varphi_P/\varphi_E| < 1$ if $\mu\eta = 1$
c) $\varepsilon \geq 1$ if $\mu\eta = 1$ and $\mu\varepsilon|\varphi_P/\varphi_E| = 1$.

Case C: $(1 - d_2^P)/(1 - d_1^P) > \gamma_1^P/\gamma_2^P, 1 < (1 - d_2^E)/(1 - d_1^E) < \gamma_1^E/\gamma_2^E$ (nonstrictly greater pursuer canard/forward control influence, strictly greater evader canard/forward control influence)

In this case the sufficient condition for the existence of a capture zone is
a) $\mu\eta > 1$
b) nonexistent if $\mu\eta = 1$.

Case D: $(1 - d_2^P)/(1 - d_1^P) < \gamma_1^P/\gamma_2^P, (1 - d_2^E)/(1 - d_1^E) > \gamma_1^E/\gamma_2^E$ (strictly greater pursuer canard/forward control influence, nonstrictly greater evader canard/forward control influence)

In this case the sufficient condition for the existence of a capture zone is
a) $\mu\eta > 1$
b) any $\mu, \varepsilon > 0$ if $\mu\eta = 1$.

Interestingly, if $\gamma_1^j = \gamma_2^j = 0.5$ then $\varphi_j = -\delta_j$. Therefore, if $\gamma_1^j = \gamma_2^j = 0.5\ \forall j \in \{P, E\}$ then $\varphi_P/\varphi_E = \eta$. In such a case, if additionally $\mu\eta = 1$ we find that $\mu\varepsilon|\varphi_P/\varphi_E| = \varepsilon$ and both conditions (b) and (c) may be replaced by $\varepsilon \geq 1$ in case A and $\varepsilon \leq 1$ in case B, respectively.

Cases A and B present the sufficient capturability conditions in an engagement between adversaries with a similar type of relative influence of their control inputs. Case C reveals that a missile without a strictly greater canard/forward control influence will be incapable of intercepting an evader with a strictly greater canard/forward control influence unless it has a distinct direct lift advantage. Case D shows that a missile with a strictly greater canard/forward control influence could be capable of capturing an evader without a strictly greater canard/forward control influence even if it does not have a distinct advantage in direct lift. These results point once again to the fact that, with respect to capturability, a greater forward control influence is more advantageous than a greater tail control influence, supporting previous conclusions [14].

In [15] the case when $0 < d_1^P < 1$, $-1 < d_2^P < 0$, $n_c^E = 1$, and $\delta_E = 0$ was examined (evader with first-order strictly proper control dynamics). In accordance with Remark IV.4, it was shown that a capture zone exists for $\delta_P \neq 0$. In addition, the presented results generalize the case analyzed in [15] to the case in which both adversaries are dual-controlled with first-order biproper dynamics.

VI. Simulations

For the sake of illustration we will now show some numerical results in the specific case of dual-controlled adversaries with first-order biproper control dynamics. We examine the following three scenarios, defined by the parameters given in Table 1.

Using the known optimal strategies in the 1-on-1 pursuit-evasion game (see [8]) we may construct the capture zone, if one exists, by backward integration of the dynamic system (21) from the final time t_f, at which $z(t_f) = 0$, to the initial time t_o. To simplify this procedure we will normalize the ZEM by $\tau_P^2 u_E^{\max}$:

$$Z(t) = \frac{z(t)}{\tau_P^2 u_E^{\max}} \tag{54}$$

The resulting nondimensional dynamic system is

$$\frac{\mathrm{d}Z}{\mathrm{d}t_{\mathrm{go}}} = \frac{1}{\tau_P^2 u_E^{\max}}\Big[f_1^P(t_{\mathrm{go}})u_1^P(t) + f_2^P(t_{\mathrm{go}})u_2^P(t) - f_1^E(t_{\mathrm{go}})u_1^E(t) - f_2^E(t_{\mathrm{go}})u_2^E(t)\Big] \tag{55}$$

for which the terminal condition of each optimal trajectory originating in the capture zone is simply $Z(t_{\mathrm{go}} = 0) = 0$. If we consider a cost function that penalizes only the miss distance $(J = |Z(t_{\mathrm{go}} = 0)|$, as is usually done, the optimal controls are easily found to be

$$\begin{aligned} \boldsymbol{u}_P^*(t) &= \mathrm{sign}\{Z(0)\}u_P^{\max}\cdot[\mathrm{sign}\{f_1^P(t_{\mathrm{go}})\}\cdot\gamma_1^P \mathrm{sign}\{f_2^P(t_{\mathrm{go}})\}\cdot\gamma_2^P]^T \\ \boldsymbol{u}_E^*(t) &= \mathrm{sign}\{Z(0)\}u_E^{\max}\cdot[\mathrm{sign}\{f_1^E(t_{\mathrm{go}})\}\cdot\gamma_1^E \mathrm{sign}\{f_2^E(t_{\mathrm{go}})\}\cdot\gamma_2^E]^T \end{aligned} \tag{56}$$

After substituting these in Eq. (61) and integrating from $Z(0)$ at time-to-go 0 (from t_f) to t_{go} (to t) we obtain the following closed form expression for the candidate optimal trajectories:

$$Z(t_{\mathrm{go}}) = Z(0) + \frac{\mathrm{sign}\{Z(0)\}}{\tau_P^2}\cdot\Big[\mu\gamma_1^P\cdot\int_0^{t_{\mathrm{go}}}|f_1^P(\theta)|\,\mathrm{d}\theta + \mu\gamma_2^P\cdot\int_0^{t_{\mathrm{go}}}|f_2^P(\theta)|\,\mathrm{d}\theta \pm \gamma_1^E\cdot\int_0^{t_{\mathrm{go}}}|f_1^E(\theta)|\,\mathrm{d}\theta - \gamma_2^E\cdot\int_0^{t_{\mathrm{go}}}|f_2^E(\theta)|\,\mathrm{d}\theta\Big]$$

Since $f_i^P(t_{\mathrm{go}})/\tau_P = f_i^P(t_{\mathrm{go}}/\tau_P)$ and $f_i^E(t_{\mathrm{go}})/\tau_P = f_i^E([t_{\mathrm{go}}/\tau_P]/\varepsilon)\ \forall i \in \{1,2\}$, we may rewrite this as a function of the normalized time-to-go t_{go}/τ_P:

$$Z(t_{\mathrm{go}}/\tau_P) = Z(0) + \mathrm{sign}\{Z(0)\}\cdot\mu\cdot\left[\gamma_1^P\cdot\int_0^{t_{\mathrm{go}}/\tau_P}|f_1^P(\theta)|\mathrm{d}\theta + \gamma_2^P\cdot\int_0^{t_{\mathrm{go}}/\tau_P}|f_2^P(\theta)|\mathrm{d}\theta\right] \pm \mathrm{sign}\{Z(0)\}\cdot\varepsilon\cdot\left[\gamma_1^E\cdot\int_0^{t_{\mathrm{go}}/\tau_P}|f_1^E(\theta/\varepsilon)|\mathrm{d}\theta + \gamma_2^E\cdot\int_0^{t_{\mathrm{go}}/\tau_P}|f_2^E(\theta/\varepsilon)|\mathrm{d}\theta\right]$$

Table 1 Simulation parameters

Parameter	Scenario 1	Scenario 2	Scenario 3
μ	1.1	$1/\eta$	$1/\eta$
ε	0.5	1.25	$1/(\mu\lvert\varphi_P/\varphi_E\rvert)$
d_2^P	−0.1	−0.1	−0.1
γ_1^P	0.5	0.45	0.25
γ_2^P	0.5	0.55	0.75
d_1^E	0.2	0.2	0.2
d_2^E	−0.1	−0.1	−0.1
γ_1^E	0.5	0.15	0.15
γ_2^E	0.5	0.85	0.85

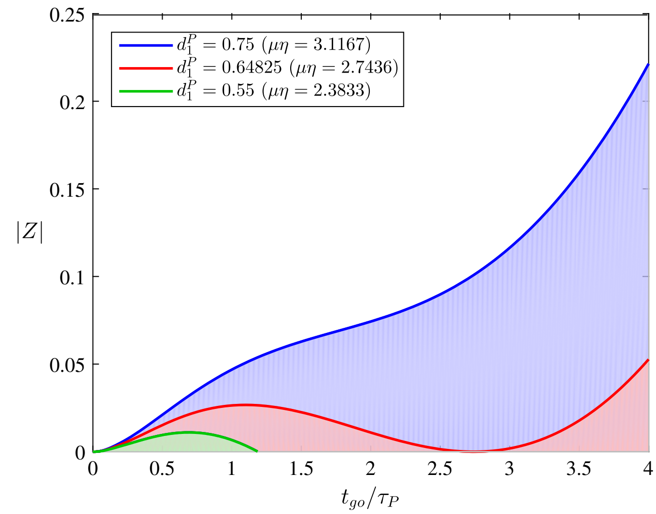

Fig. 3 Capture zone: scenario 1 ($\mu\eta > 1$).

Finally, since any couple of trajectories emanating from $Z(0)$ and $-Z(0)$ is symmetrical with respect to the $Z = 0$ axis, if one intersects this axis at some instant, so will the other. Such intersections constitute conjugate points, from which point on (in reverse time) the intersecting trajectories cease to be optimal (see Sec. 4.3.2 in [21]). Therefore, we may conclude that as long as a trajectory is optimal, $\mathrm{sign}\{Z(t_{\mathrm{go}}/\tau_P\} = \mathrm{sign}\{Z(0)\}$ and

$$|Z(t_{\mathrm{go}}/\tau_P)| = |Z(0)| + \mu\left[\gamma_1^P\cdot\int_0^{t_{\mathrm{go}}/\tau_P}|f_1^P(\theta)|\,\mathrm{d}\theta + \gamma_2^P\cdot\int_0^{t_{\mathrm{go}}/\tau_P}|f_2^P(\theta)|\,\mathrm{d}\theta\right] \pm \varepsilon\left[\gamma_1^E\cdot\int_0^{t_{\mathrm{go}}/\tau_P}|f_1^E(\theta/\varepsilon)|\,\mathrm{d}\theta + \gamma_2^E\cdot\int_0^{t_{\mathrm{go}}/\tau_P}|f_2^E(\theta/\varepsilon)|\,\mathrm{d}\theta\right] \tag{57}$$

Examples of the obtained capture zones, defined simply in terms of the normalized ZEM, are given for the different scenarios in Figs. 3–5. For each complete set of parameters the boundary of the capture zone is defined by the optimal trajectory emanating from $Z|(0)| = 0^+$ (depicted in bold). The shaded areas under these lines constitute the capture zones. Note that, since in all three scenarios $(1 - d_2^j)/(1 - d_1^j) > 1 \geq \gamma_1^j/\gamma_2^j\ \forall j \in \{P, E\}$, we refer to case B in Sec. V.E for the sufficient conditions.

In Fig. 3 several zone are shown for the first scenario when the condition $\mu\eta > 1$ is met, in accordance with condition (a) in case B. For large enough values of d_1^P we obtain an open capture zone (in blue and red), which extends, in reverse time, to all $t_{\mathrm{go}} > 0$. Otherwise, the capture zone is closed (in green), meaning that it extends only to some finite value $0 < t_{\mathrm{go}}^c < \infty$ along the $Z = 0$ axis, at which point the boundary trajectories meet to create a conjugate point.

Figure 4 shows examples of open (blue) and closed (red) capture zones in the second scenario when the condition $\mu\eta = 1$, $\mu\varepsilon|\varphi_P/\varphi_E| < 1$ is met, in accordance with condition (b) in case B.

Similarly, in Fig. 5 examples of open (blue) and closed (red) capture zones are presented for the third scenario when the condition $\mu\eta = 1$, $\mu\varepsilon|\varphi_P/\varphi_E| < 1$, $\varepsilon \geq 1$ is met, in accordance with condition (c) in case B. For the limit case in which $\varepsilon = 1$ the capture zone is reduced to $Z(t_{\mathrm{go}}) = 0\ \forall t_{\mathrm{go}} > 0$ (green). Essentially, in this case the pursuer's and the evader's maneuver capabilities are identical, and so

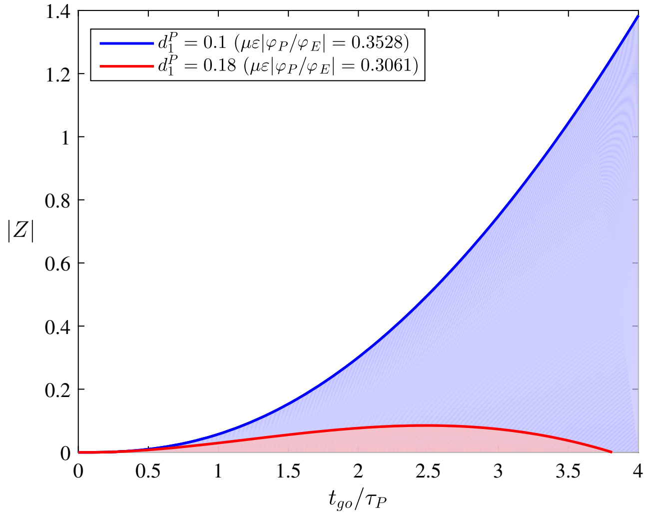

Fig. 4 Capture zone: scenario 2 ($\mu\eta = 1, \mu\varepsilon|\varphi_P/\varphi_E| < 1$).

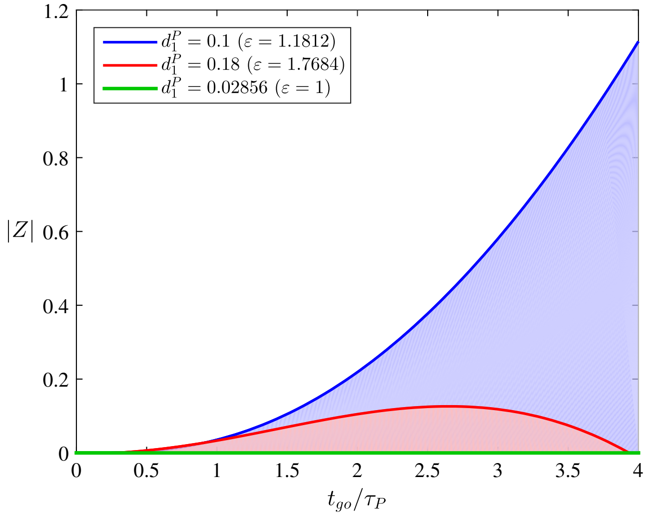

Fig. 5 Capture zone: scenario 3 ($\mu\eta = 1, \mu\varepsilon|\varphi_P/\varphi_E| = 1, \varepsilon \geq 1$).

the pursuer is able to maintain only the initial ZEM (ideally, assuming that it can react instantaneously to the evader's maneuvers). Therefore, the only initial conditions that may lead to point capture are the points along the horizontal line $Z = 0$.

VII. Conclusions

The endgame of a perfect information linear planar endoatmospheric interception engagement was examined. A new condition by which the dimension of the kinematics may be reduced was derived. It was shown that this condition extends the stricter common assumption of near head-on or tail-chase to allow the consideration of a larger variety of scenarios, in which the nominal collision triangle may be far from head-on or tail-chase.

A study of the necessary and sufficient conditions for the existence of a "hit-to-kill" capture zone in a linear interception engagement between adversaries with arbitrary-order control dynamics was presented. The existence of such a capture zone is a necessary condition for guaranteeing point capture against any target maneuver. A general condition was obtained through the solution of a linear pursuit-evasion game and an appropriate test was presented.

Based on the transfer function representation of the general control dynamics, explicit expressions of the conditions for the existence of a capture zone were derived in terms of the control dynamics characteristics of the pursuers relative to those of the evader. Such conditions are known to have been obtained in several linear pursuit-evasion games with specific adversaries' dynamics. Our results constitute an extension of the currently existing conditions to the general case of arbitrary dynamics of the adversaries.

Explicit expressions were derived from the obtained conditions for several previously studied cases. In each case the outcome was compared with known results from solutions of linear pursuit-evasion games in the same framework. This comparison was chosen because it was suggested that due to the optimality of the game solution (in particular that of the evader's strategy), any conditions for the existence of a capture zone in the game must coincide with otherwise obtained capturability conditions. Furthermore, the derived conditions included some substantial additions to what currently exists in the literature, since the developed theory enabled us to obtain the conditions for the existence of a capture zone in full, whereas in most previous studies the presented existence conditions are only partial. This is very likely because the derivation of these conditions from the game solution requires a thorough investigation of the game space structure for various combinations of the adversaries' dynamics parameters. For high-order control dynamics that include more than one or two parameters (i.e., anything beyond ideal or first-order strictly proper dynamics) this process becomes cumbersome. In comparison, the presented theory constitutes a straightforward approach to deriving the capture zone existence conditions.

An examination of the results obtained for the presented examples yielded some insight into the advantages of some dynamic properties. Mainly, they emphasize the importance of the direct lift (obtained for biproper control dynamics as opposed to strictly proper control dynamics) and the superiority of forward/canard control over tail control, supporting existing conclusions.

References

[1] Cockayne, E., "Plane Pursuit with Curvature Constraints," *SIAM Journal on Applied Mathematics*, Vol. 15, No. 6, 1967, pp. 1511–1516. doi:10.1137/0115133

[2] Isaacs, R., *Differential Games*, Wiley, New York, 1965, pp. 237–244.

[3] Rublein, G. T., "On Pursuit with Curvature Constraints," *SIAM Journal on Control*, Vol. 10, No. 1, 1972, pp. 37–39. doi:10.1137/0310003

[4] Borowko, P., and Rzymowski, W., "On the Game of Two Cars," *Journal of Optimization Theory and Applications*, Vol. 44, No. 3, 1984, pp. 381–396. doi:10.1007/BF00935458

[5] Gutman, S., and Rubinsky, S., "Exoatmospheric Thrust Vector Interception via Time-to-Go Analysis," *Journal of Guidance, Control, and Dynamics*, Vol. 39, No. 1, 2016, pp. 86–97. doi:10.2514/1.G001268

[6] Zarchan, P., *Tactical and Strategic Missile Guidance*, Vol. 157, Progress in Astronautics and Aeronautics, AIAA, Washington, D.C., 1994, Chap. 2.

[7] Gutman, S., and Leitmann, G., "Optimal Strategies in the Neighborhood of a Collision Course," *AIAA Journal*, Vol. 14, No. 9, 1976, pp. 1210–1212. doi:10.2514/3.7213

[8] Gutman, S., "On Optimal Guidance for Homing Missiles," *Journal of Guidance and Control*, Vol. 2, No. 4, 1979, pp. 296–300. doi:10.2514/3.55878

[9] Shinar, J., "Solution Techniques for Realistic Pursuit-Evasion Games," *Advances in Control and Dynamic Systems*, Vol. 17, Academic Press, New York, 1981, pp. 63–124.

[10] Shima, T., and Shinar, J., "Time-Varying Linear Pursuit-Evasion Game Models with Bounded Controls," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 3, 2002, pp. 425–432. doi:10.2514/2.4927

[11] Turetsky, V., and Shinar, J., "Missile Guidance Laws Based on Pursuit-Evasion Game Formulations," *Automatica*, Vol. 39, No. 4, 2003, pp. 607–618. doi:10.1016/S0005-1098(02)00273-X

[12] Turetsky, V., "Capture Zones of Linear Feedback Pursuer Strategies," *Automatica*, Vol. 44, No. 2, 2008, pp. 560–566. doi:10.1016/j.automatica.2007.06.014

[13] Gutman, S., "The Superiority of Canards in Homing Missiles," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 39, No. 3, 2003, pp. 740–746.
doi:10.1109/TAES.2003.1238732

[14] Shima, T., "Capture Conditions in a Pursuit-Evasion Game Between Players with Biproper Dynamics," *Journal of Optimization Theory and Applications*, Vol. 126, No. 3, 2005, pp. 503–528.
doi:10.1007/s10957-005-5495-3

[15] Shima, T., and Golan, O. M., "Bounded Differential Games Guidance Law for Dual-Controlled Missiles," *IEEE Transactions on Control Systems Technology*, Vol. 14, No. 4, 2006, pp. 719–724.
doi:10.1109/TCST.2006.876642

[16] Qi, N., Liu, Y., and Tang, Z., "Bounded Differential Game Guidance Law for Interceptor with Second-Order Maneuvering Dynamics," *Proceedings of the International Conference on Instrumentation, Measurement, Computer, Communication and Control*, IEEE Publ., Piscataway, NJ, 2011, pp. 925–928.
doi:10.1109/IMCCC.2011.233

[17] Lawrence, R. V., "Interceptor Line-of-Sight Rate Steering: Necessary Conditions for a Direct Hit," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 3, 1998, pp. 471–476.
doi:10.2514/2.4260

[18] Rusnak, I., and Meir, L., "Modern Guidance Law for High-Order Autopilot," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 5, 1991, pp. 1056–1058.
doi:10.2514/3.20749

[19] Ben-Asher, J., and Yaesh, I., *Advances in Missile Guidance Theory*, Vol. 180, Progress in Astronautics and Aeronautics, AIAA, Washington, D.C., 1998, pp. 89–126.

[20] Bryson, A. E., and Ho, Y. C., *Applied Optimal Control*, Blaisdell Publishing, Waltham, MA, 1969, Chap. 5.

[21] Glizer, V. Y., and Turetsky, V., "Complete Solution of a Differential Game with Linear Dynamics and Bounded Controls," *Applied Mathematics Research Express*, Vol. 2008, 2008, pp. 1–49, Article ID abm012.
doi:10.1093/amrx/abm012

[22] Selby, S. M., *CRC Standard Mathematical Tables*, 18th ed., Chemical Rubber Co., Cleveland, OH, 1970, p. 101.

[23] Shinar, J., Glizer, V. Y., and Turetsky, V., "The Effect of Pursuer Dynamics on the Value of Linear Pursuit-Evasion Games with Bounded Controls," *Advances in Dynamic Games, Annals of the International Society of Dynamic Games*, Vol. 13, Birkhäuser, Cham, Switzerland, 2013, pp. 313–350.
doi:10.1007/978-3-319-02690-9_15