Aircraft preforming a ground circle with presence of wind

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Introduction

This project focuses on an aircraft preforming a ground circle. The aircraft flies in the presence of wind, therefore its trajectory is circular only relatively to the ground. This project attempts to find the optimal flight energy wise.

Approaching the problem

We will approach this problem assuming the aircraft is a point of mass and assuming its thrust is equal to its drag, therefore it flies in a constant airspeed. Figure 1 describes the aircrafts flight, the symbols in it represent the following:

- v_a – The aircrafts velocity in respect to the air.
- v_g – The aircrafts velocity in respect to the ground.
- v_w – The winds velocity in respect to the ground.
- ψ – The aircrafts angle in the circle.
- ψ_a – The angle between the vector $\overrightarrow{v_a}$ and the dashed line perpendicular to the vector $\overrightarrow{v_w}$.
- L, D, T – Forces acting on the aircraft in this plane. Lift, Drag and Thrust respectively.
- ϕ – The aircrafts roll angle.

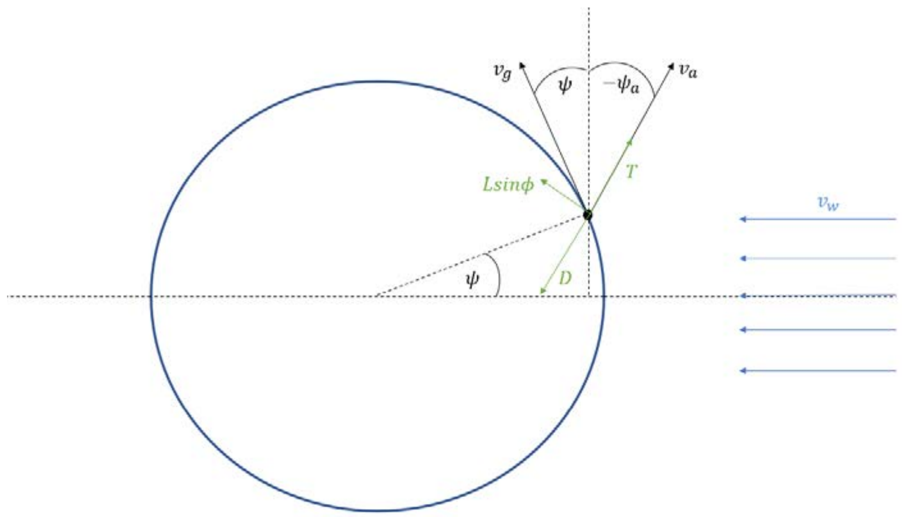


Figure 1

With ψ measured as shown, it is possible to obtain that ψ is also the angle between the dashed line perpendicular to the wind velocity vector and the aircrafts ground velocity vector (which is tangent to the preformed circle). This will be used to obtain geometrical relations.

During the analysis it will be assumed that the aircrafts velocity with respect to the air, v_a, is much larger than the winds velocity, v_w:

$$v_a \gg v_w \rightarrow \frac{v_w}{v_a} \ll 1 \qquad (1)$$

The aircrafts dynamics and leading equations

Since the thrust is equal to the drag:

$$T = D \qquad (2)$$

We can understand that the only force element contributing to the aircrafts' acceleration is $L \cdot sin\phi$. This element causes both the centripetal and tangential accelerations, I will use the centripetal equation:

$$L \cdot sin(\phi) \cdot cos(\psi - \psi_a) = \frac{W}{g} \cdot \dot{\psi} \cdot v_g \qquad (3)$$

By the definition of the roll angle, ϕ, and assuming the aircraft maintains its altitude:

$$L \cdot cos(\phi) = W \qquad (4)$$

Combining (*3*), (*4*) to get more useful expressions:

$$tan(\phi) = \frac{\dot{\psi} \cdot v_g}{g \cdot cos(\psi - \psi_a)} \qquad (5)$$

$$C_L = C_W \cdot \sqrt{1 + tan^2(\phi)} \qquad (6)$$

As briefly mentioned earlier, geometrical relations are also used in this analysis. They can help us find the dependency between the different velocities and angles. Looking at a relevant drawing will help finding these relations.

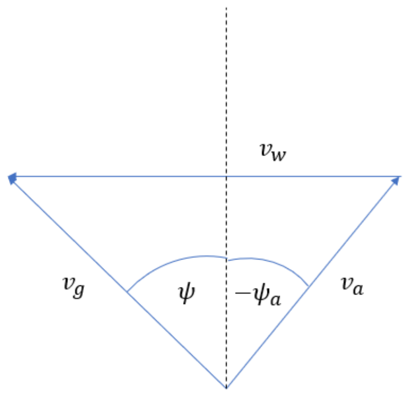


Figure 2

By trigonometric relations it is simple to obtain:

$$v_g \cdot cos(\psi) = v_a \cdot cos(\psi_a) \tag{7}$$

$$v_g \cdot sin(\psi) - v_a \cdot sin(\psi_a) = v_w \tag{8}$$

Again, making these expressions more useful is necessary:

$$\frac{v_g}{v_a} = \frac{v_w}{v_a} \cdot sin(\psi) + \sqrt{\left(\frac{v_w}{v_a}\right)^2 \cdot (sin^2(\psi) - 1) + 1} \tag{9}$$

$$cos(\psi - \psi_a) = \sqrt{\left(\frac{v_w}{v_a}\right)^2 \cdot (sin^2(\psi) - 1) + 1} \tag{10}$$

Consumed energy

The whole goal of this analysis is to minimize the consumed energy. The energy can be found by integrating the aircrafts power during a whole circle:

$$E = \int_0^T D \cdot v_a \cdot dt = \int_0^{2\pi} \frac{D \cdot v_a}{\dot{\psi}} \cdot d\psi = \int_0^{2\pi} e \cdot d\psi \tag{11}$$

Notice we are still missing the drag force, D, and the angular velocity, $\dot{\psi}$. We can find them by their definitions:

$$D = \frac{1}{2}\rho \cdot v_a^2 \cdot s \cdot (k \cdot C_L^2 + C_{D0}) \tag{12}$$

$$\dot{\psi} = \frac{v_g}{R} \tag{13}$$

The next step is to plug the equations into the energy integrand:

$$e = \frac{1}{2} \frac{s v_a^2 \rho R}{\frac{v_w}{v_a} \sin(\psi) + \sqrt{\frac{v_w}{v_a}^2 ((\sin(\psi))^2 - 1) + 1}} \left(C_W^2 \left(\frac{\left(\frac{v_w}{v_a} \sin(\ \psi) + \sqrt{\frac{v_w}{v_a}^2 ((\sin(\psi))^2 - 1) + 1}\right)^4 v_a^4}{R^2 g^2 \left(\frac{v_w}{v_a}^2 ((\sin(\psi))^2 - 1) + 1\right)} + 1 \right) k + C_{D0} \right) \tag{14}$$

The outcoming expression is very unpleasant, especially if we want to integrate it (and we do). In order to make the integration simpler, I derived the integrand into a series at $\frac{v_w}{v_a} = 0$ (recalling (*1*)). This derivation is one that can lead to many mistakes, and in order to prevent that and make the rest of the mathematical work simpler, I started using Maple. The resultant integrand, developing the series up to its 3rd term, is:

$$e = -\frac{1}{4}\frac{sv_a^2\rho}{R\,g^2}\left(\frac{v_w}{v_a}^2\left(g^2R^2\left(C_W{}^2k + C_{D0}\right) + 7\,C_W{}^2kv_a^4\right)(\cos(\psi))^2 + 2\,\frac{v_w}{v_a}\left(g^2R^2(C_W^2k + C_{D0}) - 3\,C_W^2kv_a^4\right)\sin(\psi) + \left(-2\,g^2R^2(C_W^2k + C_{D0}) - 6\,C_W^2kv_a^4\right)\frac{v_w}{v_a}^2 - 2\,g^2R^2(C_W^2k + C_{D0}) - 2\,C_W^2kv_a^4\right) + \cdots \quad (15)$$

Now, finding the energy is simple. Integrating and substituting the definition of the weight coefficient:

$$C_W = \frac{\frac{W}{s}}{\frac{1}{2}\rho v_a^2} \quad (16)$$

And the energy is:

$$E = \frac{1}{4}\frac{\rho\,v_a^2s\pi}{Rg^2}\left(12\,\frac{\frac{W^2}{s}R^2g^2k\frac{v_w}{v_a}^2}{\rho^2v_a^4} + 20\,\frac{\frac{W^2}{s}k\frac{v_w}{v_a}^2}{\rho^2} + 3\,C_{D0}\,R^2g^2\frac{v_w}{v_a}^2 + 16\,\frac{\frac{W^2}{s}R^2g^2k}{\rho^2v_a^4} + 16\,\frac{\frac{W^2}{s}k}{\rho^2} + 4\,C_{D0}\,R^2g^2\right) \quad (17)$$

I found the energy in terms of the aircrafts air velocity, and other parameters that vary between different aircrafts and circles. Having found the expression for the energy, I could now find the velocity an aircraft must fly in to minimize the energy (by finding the derivative and its roots in terms of v_a):

$$v_{a_{opt_{wind}}} = \frac{\sqrt{2}\sqrt[4]{k}\sqrt[4]{3\frac{v_w}{v_a}^2+4}\sqrt{g}\sqrt{R}\sqrt{\frac{W}{s}}}{\sqrt[4]{3\left(\frac{v_w}{v_a}^2+\frac{4}{3}\right)\rho^2g^2R^2C_{D0}+20\,k\left(\frac{v_w}{v_a}^2+\frac{4}{5}\right)\frac{W^2}{s}}} \quad (18)$$

This optimal velocity can also be found in the case of no wind:

$$v_{a_{opt_{no\,wind}}} = \frac{\sqrt{2}\sqrt[4]{k}\sqrt{g}\sqrt{R}\sqrt{\frac{W}{s}}}{\sqrt[4]{C_{D0}\,R^2g^2\rho^2+4\frac{W^2}{s}k}} \quad (19)$$

Dividing the two velocities and developing it into a series (at $\frac{v_w}{v_a} = 0$) gives us an interesting result:

$$\frac{v_{a_{opt_{wind}}}}{v_{a_{opt_{no\,wind}}}} = 1 - \frac{1}{8}\frac{1}{\frac{R^2\cdot g^2}{v^{*4}}+1}\cdot\left(\frac{v_w}{v_a}\right)^2 + O\left(\left(\frac{v_w}{v_a}\right)^4\right) \quad (20)$$

The optimal velocity with the presence of wind is slightly smaller than the optimal velocity with no wind.

We are also interested in the "price paid", the ratio between the optimal energy with wind to that without wind. In order to do so, both energies will be found, using the optimal velocities found *(18*), *(19*) and plugging them into the energy (*17)* :

$$E_{opt_{wind}} = \frac{\frac{W}{s} s\pi \sqrt{3\left(\frac{v_w}{v_a}^2+\frac{4}{3}\right)\rho^2 g^2 R^2 C_{D0}+20\,k\left(\frac{v_w}{v_a}^2+\frac{4}{5}\right)\frac{W^2}{s}}\,\sqrt{k}\sqrt{3\frac{v_w}{v_a}^2+4}}{\rho\, g} \tag{21}$$

$$E_{opt_{no\ wind}} = 4\,\frac{\frac{W}{s}\pi\, s\sqrt{C_{D0}R^2 g^2\rho^2+4\frac{W^2}{s}k}\sqrt{k}}{\rho\, g} \tag{22}$$

Therefore, the price is:

$$p = \frac{E_{opt_{wind}}}{E_{opt_{no\ wind}}} = \frac{1}{4}\,\frac{\sqrt{3\left(\frac{v_w}{v_a}^2+\frac{4}{3}\right)\rho^2 g^2 R^2 C_{D0}+20\,k\left(\frac{v_w}{v_a}^2+\frac{4}{5}\right)\frac{W^2}{s}}\,\sqrt{3\frac{v_w}{v_a}^2+4}}{\sqrt{C_{D0}\,R^2 g^2\rho^2+4\frac{W^2}{s}k}} \tag{23}$$

Developing this price to a series at $\frac{v_w}{v_a} = 0$:

$$p = 1 + \frac{\left(3\,C_{D0}\,R^2 g^2\rho^2+16\frac{W^2}{s}k\right)\frac{v_w}{v_a}^2}{4\,C_{D0}\,R^2 g^2\rho^2+16\frac{W^2}{s}k} + O\left(\frac{v_w}{v_a}^4\right) \tag{24}$$

Now we can address to extreme cases:

1) The circle goes to infinity:

$$p_{R\to\infty} = 1 + \frac{3}{4}\left(\frac{v_w}{v_a}\right)^2 + O\left(\frac{v_w}{v_a}^4\right) \tag{25}$$

2) The circle is very small:

$$p_{R\to 0} = 1 + \left(\frac{v_w}{v_a}\right)^2 + O\left(\frac{v_w}{v_a}^4\right) \tag{26}$$

In both cases the price is, as expected, larger than 1, however it is very small. The wind is taking its toll but in a very modest way.

Optimization parameters

In this problem I optimized the consumed energy. I did so by expressing all the non-constant flight parameters with the aircrafts' velocity and finding the optimal one. Therefore, finding this optimal velocity determines these non-constant flight parameters, the optimization parameters.

Most parameters will be described in a non-dimensional manner. I normalized each parameter to make it non-dimensional and expressed it with some non-dimensional quantities.

A non-dimensional quantity that found its way to all the optimization parameters expressions and has a physical meaning is:

$$G = \frac{(v^*)^2}{Rg} \tag{27}$$

While:

$$v^* = \sqrt{\frac{2W}{\rho S C_L^*}} \tag{28}$$

And:

$$C_L^* = \sqrt{\frac{C_{D0}}{k}} \tag{29}$$

G describes the normalized centripetal acceleration with no wind travelling at v^*. Considering an aircraft preforming a regular circle without wind, G will be equal to the tangent of the roll angle. Using equation (*3*) for the no wind case ($\psi = \psi_a$) and substituting (*4*), (*13*) in:

$$\frac{v^2}{Rg} = tan(\phi) \tag{30}$$

While v^* is used instead of the regular velocity in order to make the analysis generic.

- The velocity: The velocity is the parameter we used in order to find the optimal energy. In order to see how it changes throughout the circle, we need to manipulate equation (*18*). Initially, multiplying and dividing the RHS by $\sqrt{\sqrt{k} \cdot \frac{W}{S}}$:

$$v_{a_{opt_{wind}}} = \frac{\sqrt{2}\sqrt[4]{3\frac{v_w{}^2}{v_a}+4\sqrt{g}\sqrt{R}}}{\sqrt[4]{3\left(\frac{v_w{}^2}{v_a}+\frac{4}{3}\right)\frac{\rho^2 g^2 R^2 C_{D0}}{k\left(\frac{W}{S}\right)^2}+20\left(\frac{v_w{}^2}{v_a}+\frac{4}{5}\right)}} \tag{31}$$

Substituting (*27*), (*28*) and (*29*) in and dividing both sides by v^*:

$$\frac{v_{a_{opt_{wind}}}}{v^*} = \frac{\sqrt{2}\sqrt[4]{3\frac{v_w{}^2}{v_a}+4\sqrt{1/G}}}{\sqrt[4]{3\left(\frac{v_w{}^2}{v_a}+\frac{4}{3}\right)4\left(\frac{1}{G}\right)^2+20\left(\frac{v_w{}^2}{v_a}+\frac{4}{5}\right)}}$$

This expression is non dimensional and is only dependent in two non-dimensional quantities. One can now see how the optimal (normalized) velocity changes with respect to the wind and the circle accelerations preformed:

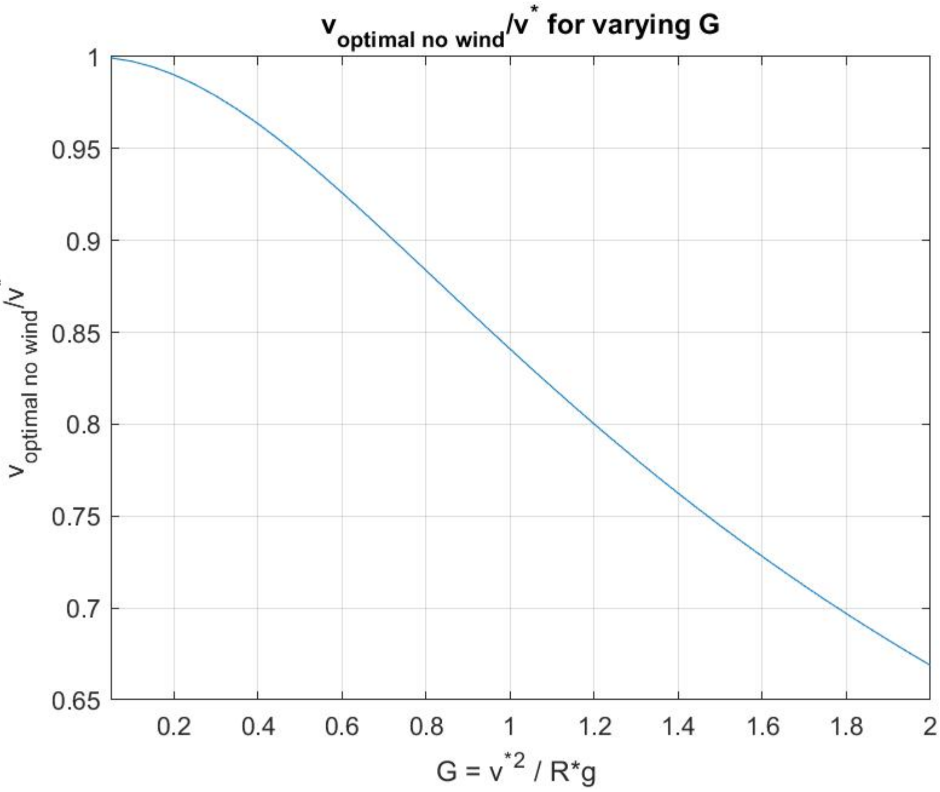


Figure 3

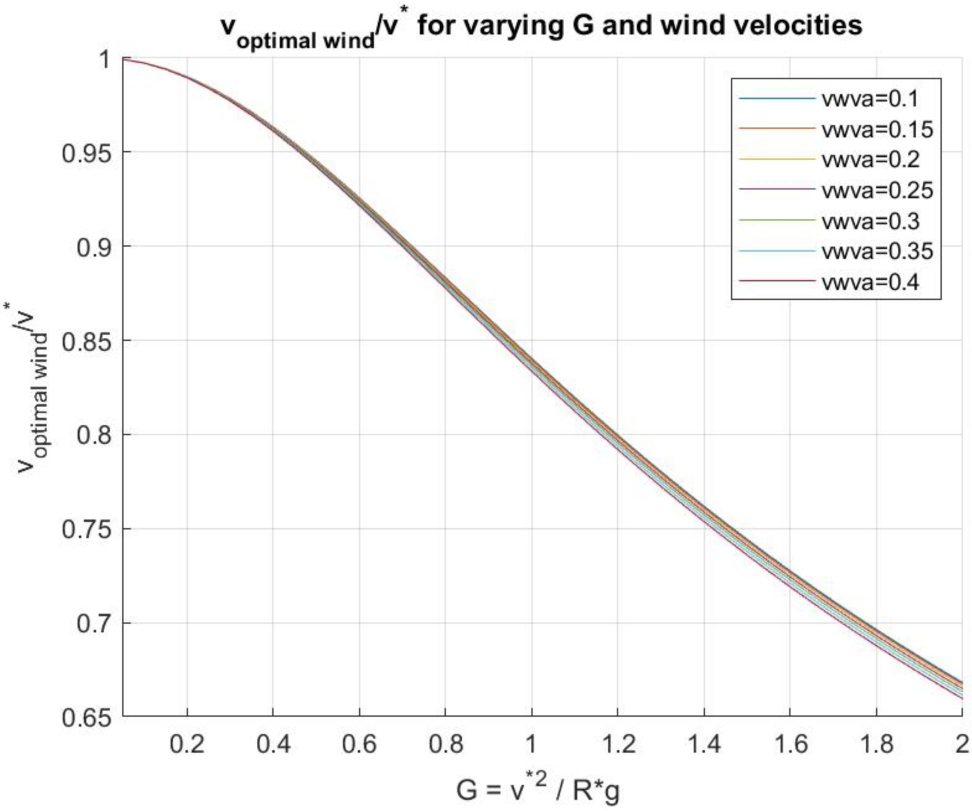


Figure 4

As shown, the optimal velocity was found with and without the presence wind. The wind does not seem to affect the optimal velocity too much in this case. The ratio between the two will tell us how big a change a pilot or a control system must make in order to maintain an optimal circle.

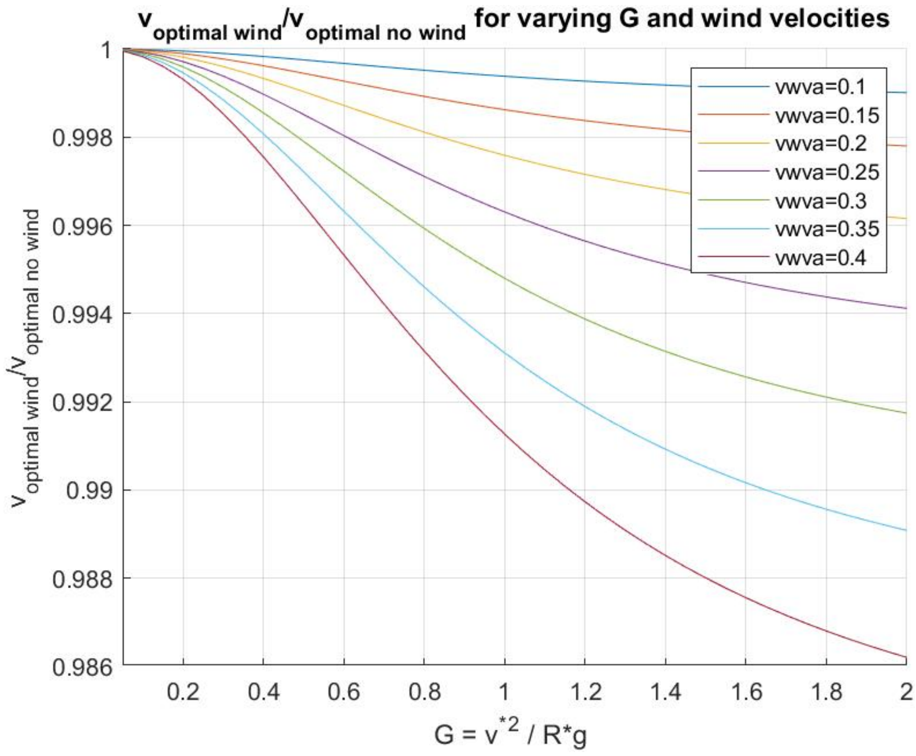


Figure 5

- The lift coefficient: In this problem the aircraft maintains constant altitude. Therefore, the vertical lift component must balance the weight. However, this component changes as the roll angle changes throughout the circle (as we will see later), thus the lift coefficient changes as well.
  In order to find a suitable expression for the lift coefficient I plugged equations (*5*), (*9*), (*10*), (*13*) and (16) into equation (*6*):

$$C_L = 2\,\frac{W/s}{\rho\, v_a^2}\sqrt{\frac{\left(\frac{v_w}{v_a}\,sin(\psi)+\sqrt{\frac{v_w}{v_a}^2((sin(\psi))^2-1)+1}\right)^4 v_a^4}{\left(\frac{v_w}{v_a}^2((sin(\psi))^2-1)+1\right)R^2 g^2}+1} \quad (32)$$

$$C_L = 2\,\frac{\frac{W}{s}}{\rho\, v_a^2}\frac{(v^*)^2}{(v^*)^2}\sqrt{\frac{\left(\frac{v_w}{v_a}\,sin(\psi)+\sqrt{\frac{v_w}{v_a}^2((sin(\psi))^2-1)+1}\right)^4 v_a^4}{\left(\frac{v_w}{v_a}^2((sin(\psi))^2-1)+1\right)R^2 g^2}\frac{(v^*)^4}{(v^*)^4}+1} \quad (33)$$

Using (*28*) and (*27*):

$$\frac{C_L}{C_L^*} = \frac{1}{\frac{v_a^2}{(v^*)^2}}\sqrt{\frac{\left(\frac{v_w}{v_a}\,sin(\psi)+\sqrt{\frac{v_w}{v_a}^2((sin(\psi))^2-1)+1}\right)^4}{\left(\frac{v_w}{v_a}^2((sin(\psi))^2-1)+1\right)}\,G^2\left(\frac{v_a}{v^*}\right)^4+1} \quad (34)$$

This expression is now non-dimensional and uses only 3 non dimensional numbers – only 2 of them are nondependent. As shown, $\frac{v_a}{v^*}$ is a function of G and $\frac{v_w}{v_a}$. With this expression obtained I could find different (normalized) lift coefficients for different wind speeds and different circle accelerations preformed.

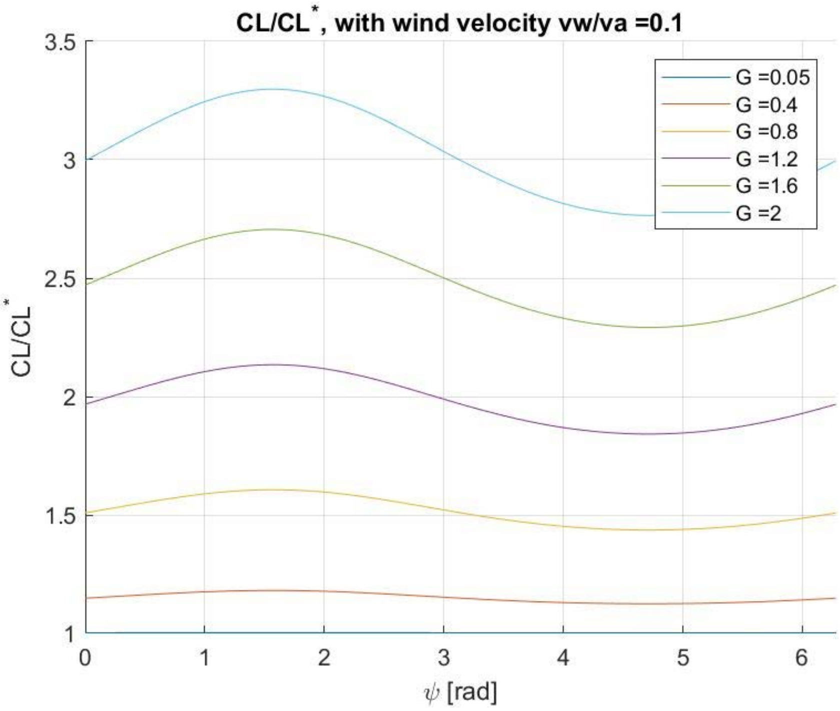


Figure 6

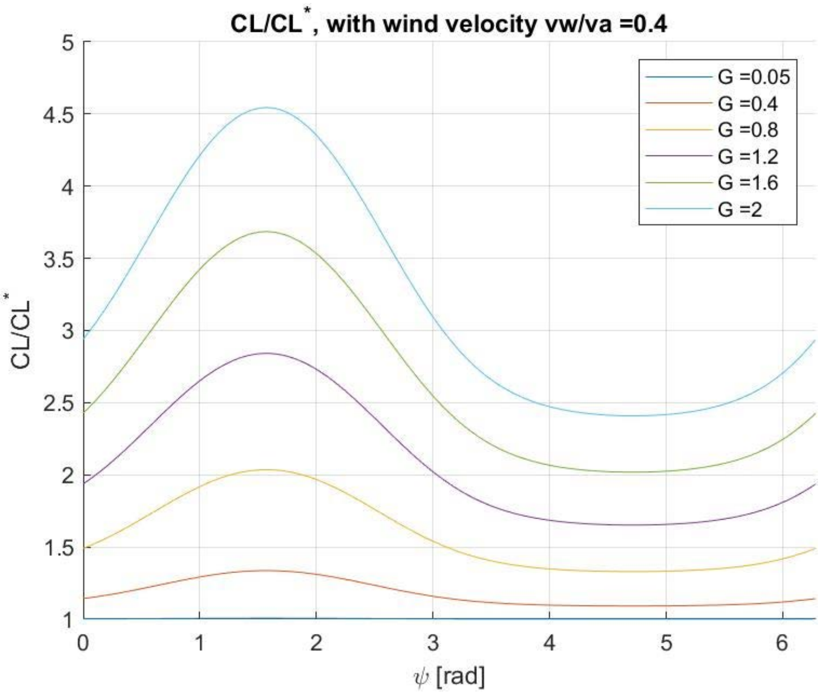


Figure 7

The optimal lift coefficient was found for optimal circles with wind and without wind. This figure describes the ratio between the lift coefficient with wind to that without, it describes the change a pilot or a control system will have to perform in a case of a gust of wind coming by.

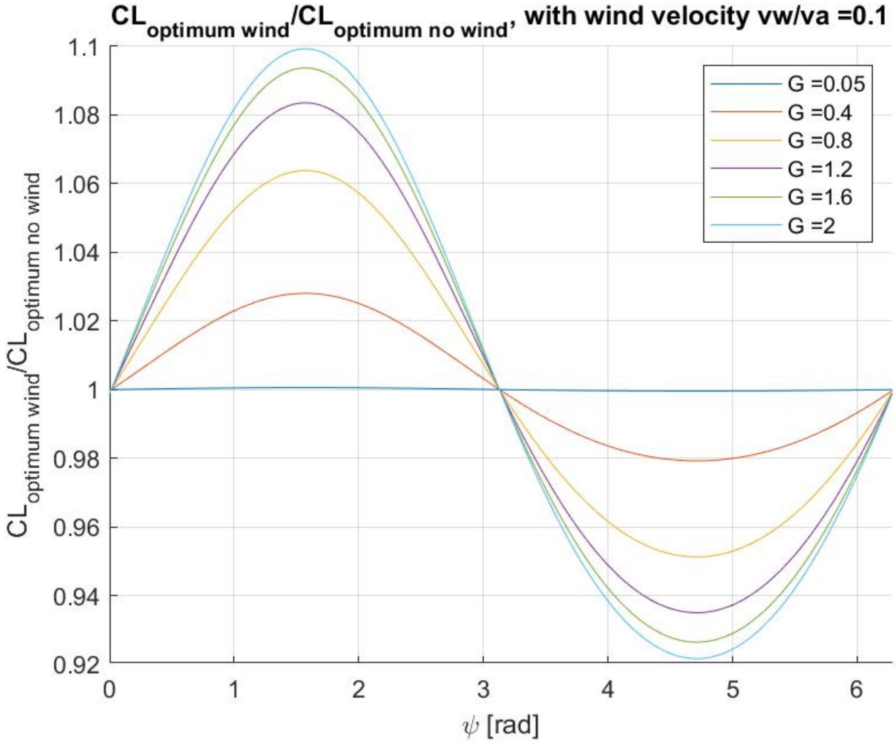


Figure 8

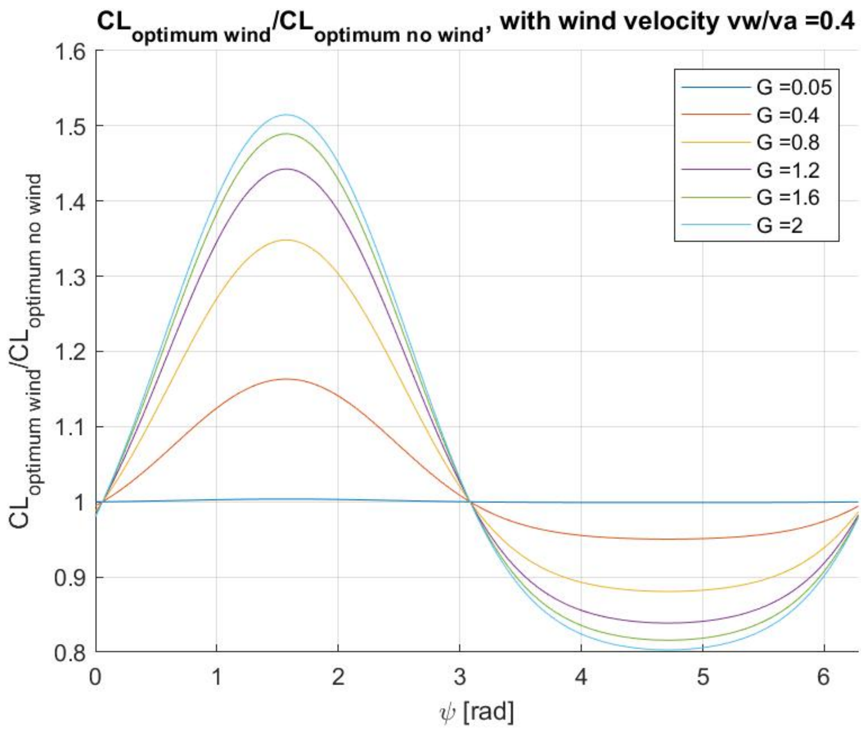


Figure 9

Aircrafts have a maximal lift coefficient that can be smaller than the optimal lift coefficient for certain cases. In this situation the aircraft will have to remain at the smaller lift coefficient. In case like this the aircraft will not be able to perform the desired circle.

- The load factor/roll angle: Instead of showing these two parameters I will show only the roll angle remembering that $n = \frac{1}{\cos(\phi)}$. In order to find a suitable expression for the tilt angle I plugged equations (*9*), (*10*) and (*13*) into equation (*5*):

$$tan(\phi) = \frac{\left(\frac{v_w}{v_a} sin(\psi)+\sqrt{\frac{v_w}{v_a}^2((sin(\psi))^2-1)+1}\right)^2 v_a^2}{\sqrt{\frac{v_w}{v_a}^2((sin(\psi))^2-1)+1}\, Rg} \quad (35)$$

Using (*27*):

$$tan(\phi) = \frac{\left(\frac{v_w}{v_a} sin(\psi)+\sqrt{\frac{v_w}{v_a}^2((sin(\psi))^2-1)+1}\right)^2}{\sqrt{\frac{v_w}{v_a}^2((sin(\psi))^2-1)+1}} G\left(\frac{v_a}{v^*}\right)^2 \quad (36)$$

For different wind speeds, the optimal roll angle can now be found for chosen G values, as shown here:

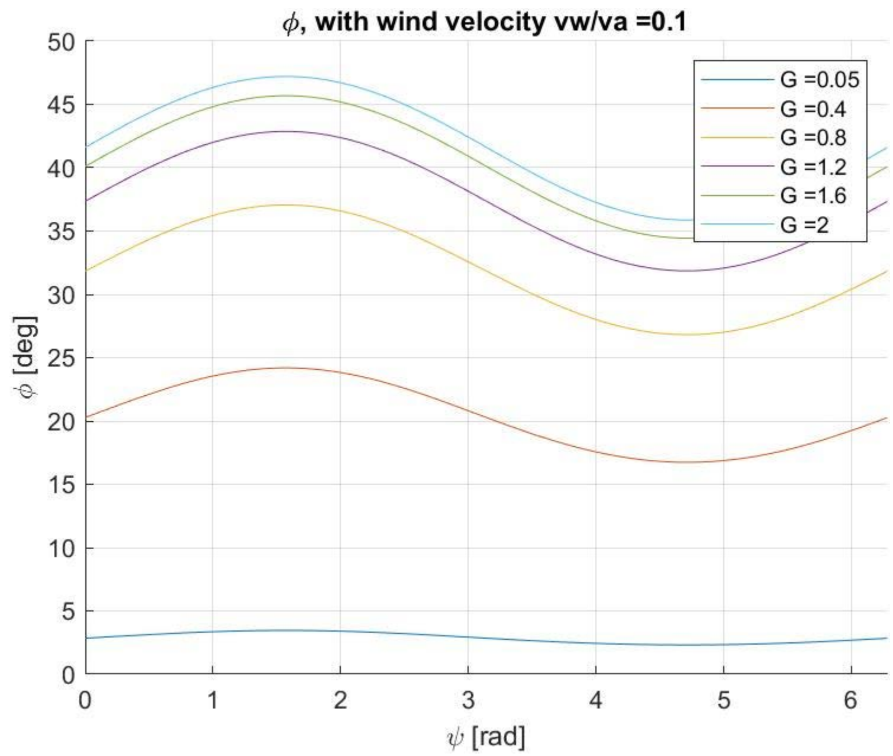


Figure 10

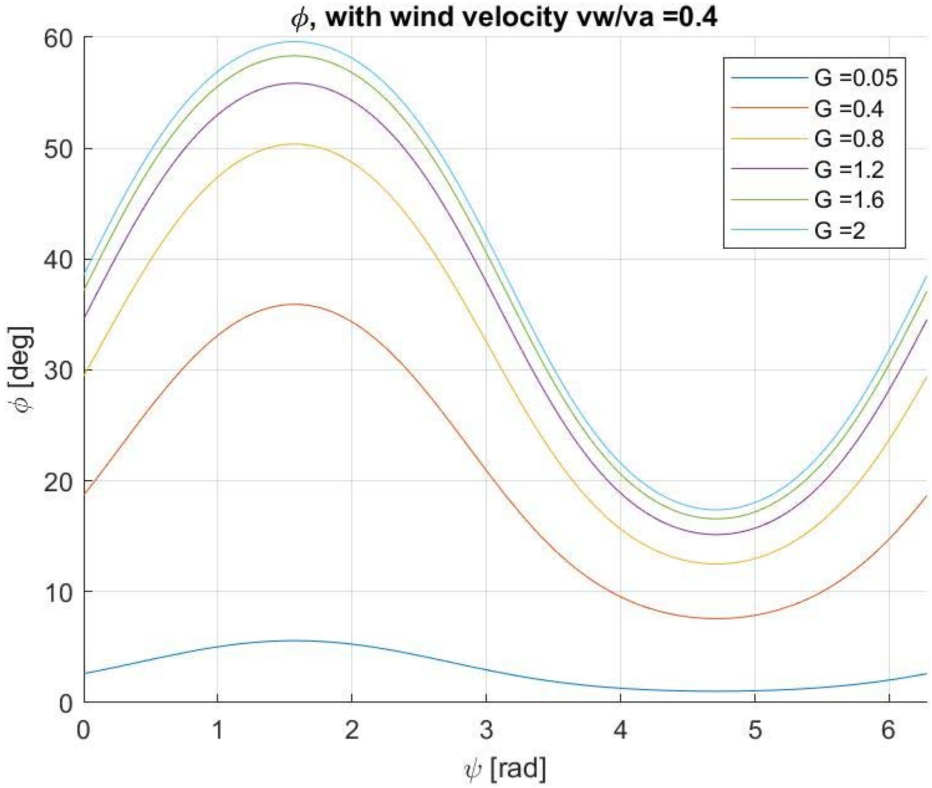


Figure 11

As done earlier, the optimal roll angle was found with wind and without wind. The no wind optimal roll angle is obviously constant. The ratio between the optimal roll angle with wind to that without, will show us, yet again, what changes a pilot or control system will have to implement.

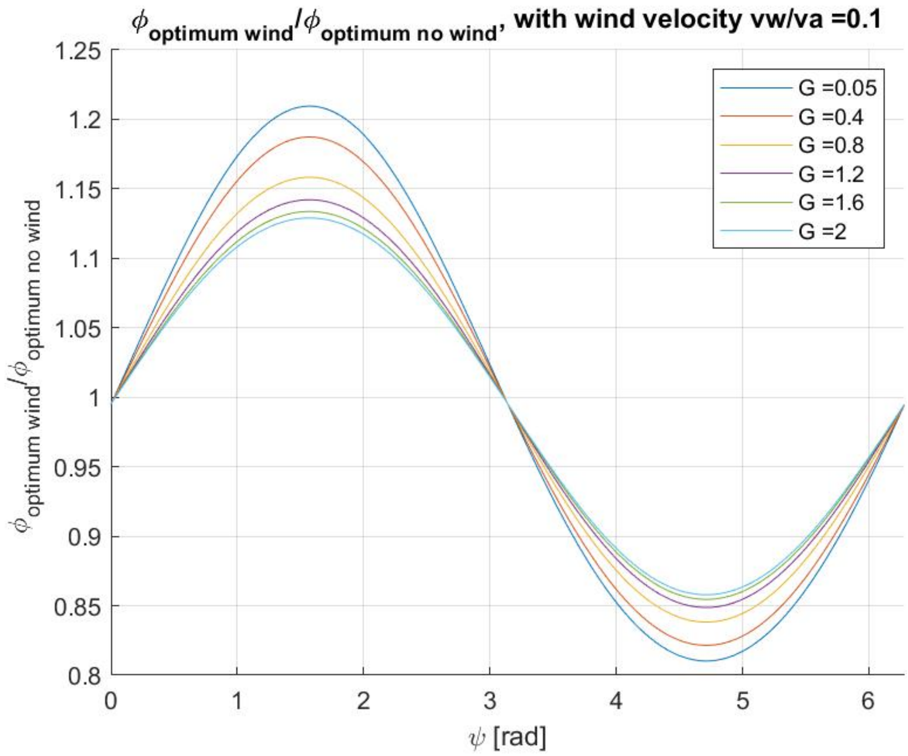
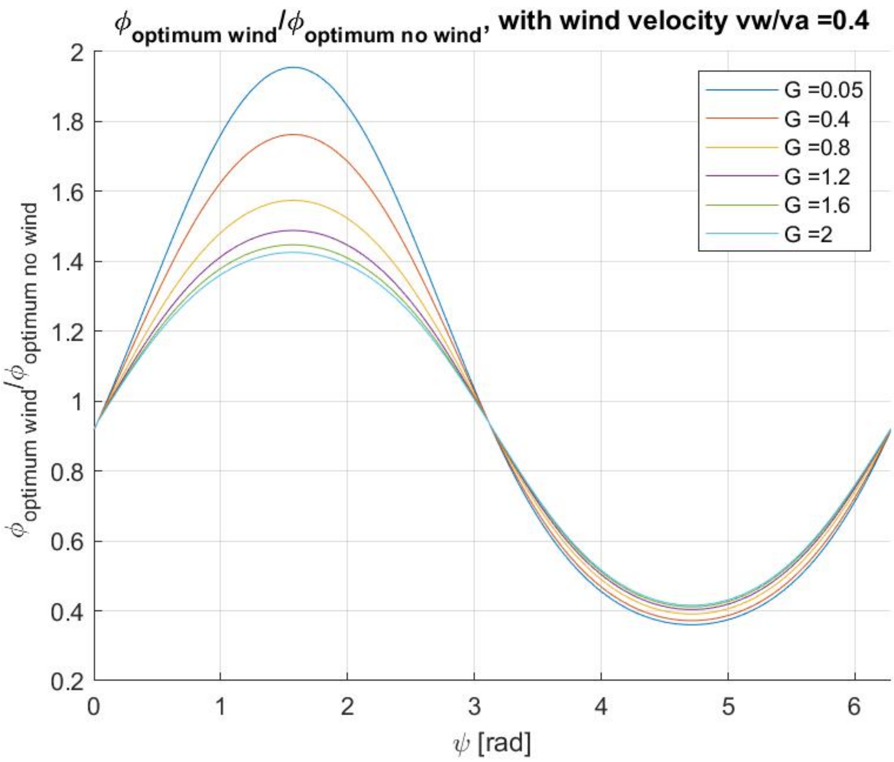


Figure 12



$\phi_{optimum\ wind}/\phi_{optimum\ no\ wind}$, with wind velocity vw/va =0.4
G =0.05
G =0.4
G =0.8
G =1.2
G =1.6
G =2
$\phi_{optimum\ wind}/\phi_{optimum\ no\ wind}$
ψ [rad]

Figure 13

- <u>The price</u>: The price, found in (*23*), is obviously another interesting parameter to observe. Dividing and multiplying the RHS by $\sqrt{\left(\frac{W}{s}\right)^2 k}$ and applying (*27*), (*28*) and (*29*):

$$p = \frac{1}{8}\frac{\sqrt{3\left(\frac{v_w^2}{v_a}+\frac{4}{3}\right)4\left(\frac{1}{G}\right)^2+20\left(\frac{v_w^2}{v_a}+\frac{4}{5}\right)}\sqrt{3\frac{v_w^2}{v_a}+4}}{\sqrt{\left(\frac{1}{G}\right)^2+1}} \qquad (37)$$

Now, the price can be found for different wind speeds and different circle accelerations:

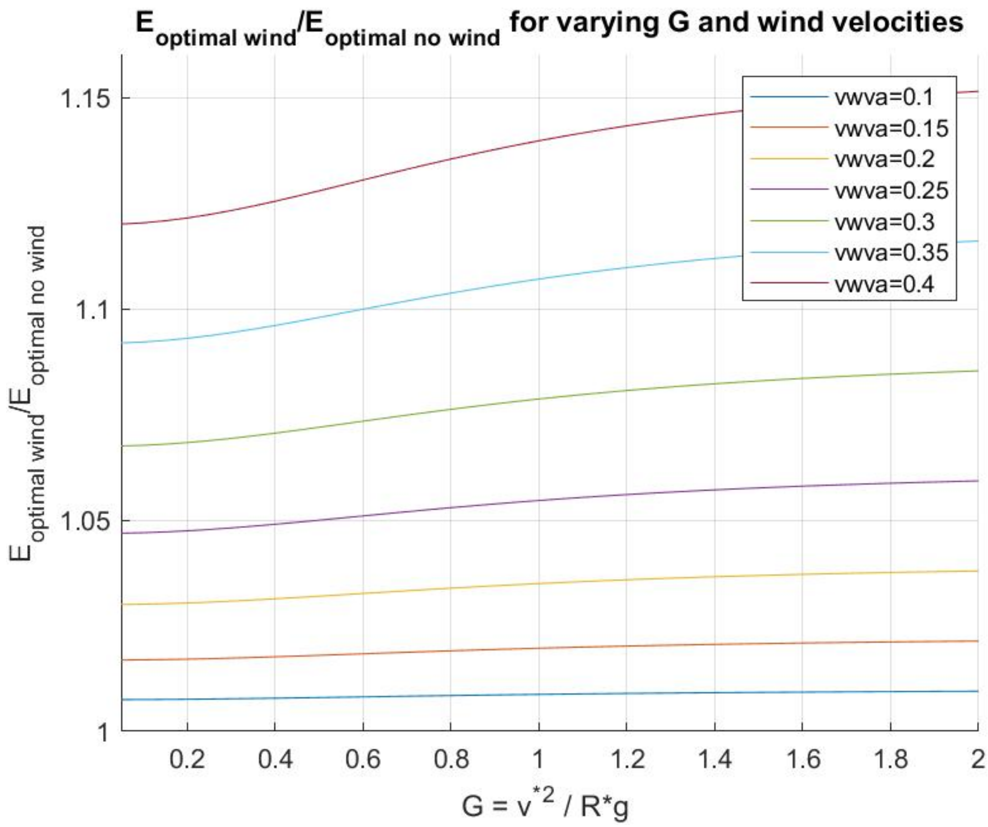


Figure 14

Conclusions

In this work we tried to find the optimal constant velocity (in respect to the air) an aircraft must fly in to minimize its energy usage. We found that the optimal velocity is smaller and very close to the velocity in the case where there is no wind, and the price paid is very low. We will try to understand these findings:

1. Why is $v_{a_{opt}}{}_{wind} < v_{a_{opt}}{}_{no\ wind}$?

This can occur since we are trying to take advantage of the wind. It makes sense that both ground velocities are close so perhaps flying at a slightly smaller velocity helps us more on the back-wind half than it "hurts" us on the nose wind half.

2. Why is the price paid so low?

As we can see in Figure 14, the price becomes a factor to consider only when the wind is very strong. This is probably a consequent of the optimal velocities being so close. If the energy was only a function of the velocities, one would expect that the price would even be smaller than 1. However, there are other factors, such as the winds velocity, that affects the energy integrand as we saw in (14). These factors apparently make the price slightly larger than 1. Observing the expression for the energy (*17)* makes things clearer. The energy does grow with the presence of wind.

During the work we assumed that the thrust is equal to the drag, therefore $v_a = const$. It will be interesting to see what happens when we get rid of this constraint and try to find the optimal v_a as a function of the location at the circle and the wind direction. Will we be able to improve the price? Can we take advantage of the wind?