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Modelling of Tonal Noise Component in Propellers

מידול רכיב רעש טונאלי במדחפים

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1 Background and Introduction

In recent years, due to technological advancements and the decrease in manufacturing costs, the field of unmanned aerial vehicles (UAV) is in rapid growth. Most of UAV are propelled by a rotating wing popustion systems. An unwanted by product of using a rotating wing propulsion system is a fairly high noise level, and while most of UAV are planned to operate at low altitudes and frequently at civil environment, this issue takes priority in design and development. One of the main sources to this high noise levels is the unsteady forces acting on the propeller. At hovering conditions, the incoming flow consists of vortex sheets detached from the blade itself and from other blades in propeller that generates a highly unsteady flow regime. This unsteadiness takes form of an unsteady force acting on the blade, and due to the periodic nature of the rotating propeller at hovering conditions, this force also takes a periodic form.

This project is focusing on modeling the tonal noise component driven by both quasi-steady and unsteady forces of a propeller at hover in far-field conditions, using a frequency domain analysis to decompose the noise levels into harmonies and including the unsteady vortex sheets effect by a lift deficiency function. Furthermore, an approximated spanwise force and chord distribution function are suggested and effect of each distribution is reviewed and functions of this type are suggested as an acoustic design parameter.

2 Governing Equations

2.1 Lighthill's Analogy

In order to model tonal noise components, first we derive the governing equation under two main assumptions, acoustical far-field $R \gg \lambda$ and geometrical far-field $R \gg r'$, where R is the distance from source, λ is the acoustical wavelength and r' is the characteristic length of the source.

Define small perturbations around mean pressure

$$p = p_0 + p' \tag{1}$$

and density

$$\rho = \rho_0 + \rho' \tag{2}$$

where $_0$ indicates mean quantity and $'$ is the perturbation.

For isentropic flow the relation between the two perturbations presented above is

$$p' = \rho' c_0^2 \tag{3}$$

where c_0 is the speed of sound in standard conditions. Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \tag{4}$$

Momentum consevation equation:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \left(\rho u_i u_j + p_{ij}\right)}{\partial x_i} - F_i = 0 \tag{5}$$

where F_i is an external force and $p_{i,j} = (p - p_0)\delta_{ij} - \sigma_{ij}$ are viscous stress tensor and pressure perturbation contribution. Using conservation Eqs. (4) and (5), we can derive Lighthill's analogy [1]. Derive the continuity equation (4) by time:

$$\frac{\partial}{\partial t}\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i}\right) = 0 \tag{6}$$

Take the divergence of the momentum Eq. (5):

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \rho u_i}{\partial t} + \frac{\partial\left(\rho u_i u_j + p_{ij}\right)}{\partial x_i} - F_i\right) = 0 \tag{7}$$

Subtract Eq. (6) from Eq. (7):

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho u_i u_j + (p - p_0)\delta_{ij} - \sigma_{ij}\right) - \frac{\partial F_i}{\partial xi} \tag{8}$$

Subtract the Lapalacian of the pressure perturbation in terms of Eq. (3) from both sides of Eq. (8) and obtain:

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho u_i u_j + (p' - c_0^2 \rho')\delta_{ij} - \sigma_{ij}\right) - \frac{\partial F_i}{\partial xi} \tag{9}$$

Assuming that ρ_0 is constant in time we can derive the final form of Lighthill's wave equation:

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} - \frac{\partial F_i}{\partial xi} \tag{10}$$

where $T_{ij} = \rho u_i u_j + (p' - c_0^2 \rho')\delta_{ij} - \sigma_{ij}$ is Lighthill's stress tensor and represents Reynolds stresses, the effect of pressure and density perturbations and viscous stress tensor. Assuming acoustic wavelength based Reynolds number, $Re = \frac{c_0 \lambda}{\nu}$, means that the inertial forces are much greater than the viscous related terms, therefore it can be neglected from Lighthill's tensor. Furthermore, the term $(p' - c_0^2 \rho')$ is small in subsonic flows thus can be neglected as well.

2.2 Free Space Green's Function - Solution to Lighthill's Wave Equation

The commonly known solution to Lighthill's wave equation is obtained by the free space Green's function which represent a point source with a strength $\delta(\mathbf{x} - \mathbf{y})\delta(t - \tau)$:

$$G = \frac{\delta\left(t - \tau - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{4\pi c_0^2 |\mathbf{x} - \mathbf{y}|} \tag{11}$$

where $\mathbf{x}$ is an observer, $\mathbf{y}$ is a source and τ is retarded time.

So the solution to Lighthill's wave equation Eq. (10) is:

$$\rho'(\mathbf{x}, t) = \int_t \int_V \left(\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}\right) G(\mathbf{y}, t - \tau) d\tau dV - \int_t \int_S \left(\frac{\partial F_i}{\partial xi}\right) G(\mathbf{y}, t - \tau) d\tau dS =$$
$$\int_t \int_V \left(\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}\right) \frac{\delta\left(t - \tau - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{4\pi c_0^2 |\mathbf{x} - \mathbf{y}|} d\tau dV - \int_t \int_S \left(\frac{\partial F_i}{\partial xi}\right) \frac{\delta\left(t - \tau - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{4\pi c_0^2 |\mathbf{x} - \mathbf{y}|} d\tau dS \tag{12}$$

Recall the relation between pressure perturbation and density Eq. (3) and notice that the Eq. (12) contains a convolution integral:

$$p'(\mathbf{x},t) = \frac{1}{4\pi}\int_V \frac{\partial^2}{\partial x_i \partial x_j} T_{ij}\left(y, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)\frac{1}{|\mathbf{x}-\mathbf{y}|}dV - \frac{1}{4\pi}\int_S \frac{\partial}{\partial x_i} F_i\left(y, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)\frac{1}{|\mathbf{x}-\mathbf{y}|}dS \quad (13)$$

A rotating propeller can be modelled as a dipole, thus the pressure perturbation for a this case is:

$$p'(\mathbf{x},t) = -\frac{1}{4\pi}\int_S \frac{\partial}{\partial x_i} F_i\left(y, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)\frac{1}{|\mathbf{x}-\mathbf{y}|}dS \quad (14)$$

2.3 Far-Field Approximation

Recall the assumption of geometrical far-field, hence $|\mathbf{x}/\mathbf{y}| \ll 1$, thus:

$$|\mathbf{x}-\mathbf{y}| = \left(|\mathbf{x}|^2 - 2\mathbf{x}\mathbf{y} + |\mathbf{y}|^2\right)^{1/2} = |\mathbf{x}|\left(1 - \frac{2\mathbf{x}\cdot\mathbf{y}}{\mathbf{x}^2} + \frac{\mathbf{y}^2}{\mathbf{x}^2}\right)^{1/2} \approx |\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} \quad (15)$$

Furthermore:

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} = \frac{1}{|\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}} \approx \frac{1}{|\mathbf{x}|}\left(1 + \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right) \approx \frac{1}{|\mathbf{x}|} \quad (16)$$

Notice that the integrand in Eq. (14) can be simplified:

$$\frac{\partial}{\partial x_i} F_i\left(y, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right) = \frac{\partial \tau}{\partial x_i}\frac{\partial}{\partial \tau} F_i\left(y, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right) = -\frac{1}{Dc_0}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\frac{\partial F_i}{\partial \tau} \quad (17)$$

where $D = 1 - M\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}$ is Doppler correction.

Thus the pressure perturbation under far-field approximation takes the form of:

$$p'(\mathbf{x},t) = \frac{1}{4\pi c_0}\int_S \frac{R_i}{DR^2}\frac{\partial}{\partial \tau} F_i\left(y, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right) dS \quad (18)$$

where $R_i = x_i - y_i$ and $R = |\mathbf{x}-\mathbf{y}|$.

In order to keep simplifying Eq. (18), assume the source is compact, i.e. $He = R/\lambda \ll 1$ where λ is the acoustic wavelength, then the S collapse into a point so Eq. (18) can be written as:

$$p'(\mathbf{x},t) = \frac{R_i}{4\pi c_0 DR^2}\left[\frac{\partial F}{\partial \tau}\right] \quad (19)$$

where $\left[\frac{\partial F}{\partial \tau}\right]$ is the unsteady forces acting on fluid by the rotating blade.

2.4 Frequency Domain Analysis

Recall that the dipole source is a propeller rotating at angular velocity of Ω, thus it holds a periodic nature. In order to find the contribution of each harmony a Fourier Transform is needed to be done in terms of Ω.

$$p'(\mathbf{x},\Omega) = \frac{\Omega}{2\pi}\int_{-\infty}^{\infty} \frac{R_i}{4\pi c_0 DR^2}\left[\frac{\partial F}{\partial \tau}\right] e^{i\Omega t}dt \quad (20)$$

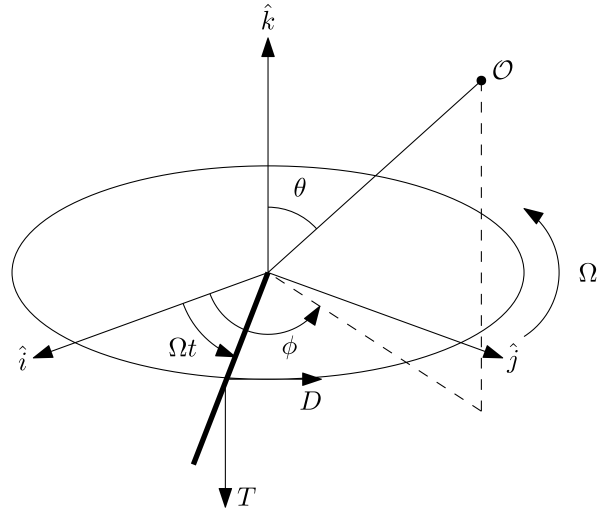


Figure 1: rotating blade & observer coordinate system and definitions.

The limits of the integral in Eq. (20) can be adapted to fit the periodic nature of the physical system:

$$p'(\mathbf{x},\Omega) = \frac{\Omega}{8\pi^2 c_0}\int_0^{2\pi/\Omega} \frac{R_i}{DR^2}\left[\frac{\partial F}{\partial \tau}\right] e^{i\Omega t} dt \tag{21}$$

Applying the relation between retarded time derivative and time derivative $\frac{\partial t}{\partial \tau} = D$:

$$p'(\mathbf{x},\Omega) = \frac{\Omega}{8\pi^2 c_0}\int_0^{2\pi/\Omega} \frac{R_i}{R^2}\left[\frac{\partial F}{\partial \tau}\right] e^{i\Omega(\tau+R/c_0)} d\tau \tag{22}$$

In order to eliminate the partial derivative in Eq. (22) an integration by parts is used:

$$p'(\mathbf{x},\Omega) = \frac{\Omega}{8\pi^2 c_0}\left[\frac{R_i}{R^2} F e^{i\Omega(\tau+R/c_0)}\Big|_0^{2\pi/\Omega} - \int_0^{2\pi/\Omega} \frac{R_i}{R^2} i\Omega F e^{i\Omega(\tau+R/c_0)} d\tau\right]$$

So the Fourier transform of the pressure perturbation is:

$$p'(\mathbf{x},\Omega) = -\frac{i\Omega^2}{8\pi^2 c_0}\int_0^{2\pi/\Omega} \frac{R_i}{R^2} F e^{i\Omega(\tau+R/c_0)} d\tau \tag{23}$$

Because of the periodic nature of the source, the pressure perturbation felt by the observer must be a multiple of the rotation frequency of the source. Define n-th harmonics as the n-th multiple of the pressure perturbation created by the source:

$$C_n = -\frac{ni\Omega^2}{8\pi^2 c_0}\int_0^{2\pi/\Omega} \frac{R_i}{R^2} F e^{i\Omega(\tau+R/c_0)} d\tau \tag{24}$$

2.5 The Physical System

Assume a rotating blade with an angular velocity of Ω, where the center of rotation is at the origin and x_1, x_2 is the rotation plane. The force acting on the fluid by the blade is composed by T thrust component and D drag component. Assume an observer $\mathcal{O}$, located at a distance l from the origin, at an angle θ from axis of rotation $\hat{k}$ and at an angle ϕ on the rotation plane x_1, x_2. So according to Fig. 1:

$$\mathbf{x} \equiv \mathcal{O} = l\left[\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}\right]$$

$$F = -D\sin\Omega t\hat{i} + D\cos\Omega t\hat{j} - T\hat{k}$$

$$\mathbf{y} = r\left[\cos\Omega t\hat{i} + \sin\Omega t\hat{j} + 0\hat{k}\right]$$

2.6 n-th Harmonic Pressure Perturbation

Applying the physical system definitions as presented in Fig. 1 on the appropriate terms in Eq. (24):

$$R_i = \mathbf{x} - \mathbf{y} = (l\sin\theta\cos\phi - r\cos\Omega t)\hat{i} + (l\sin\theta\sin\phi - r\sin\Omega t)\hat{j} + l\cos\theta\hat{k}$$

then

$$R_i F = Dl\sin\theta\sin(\phi - \Omega t) - Tl\cos\theta$$

by applying far-field approximation Eqs. (15)-(16):

$$R = |\mathbf{x} - \mathbf{y}| \approx \left(|\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right) \approx l\left(1 - \frac{r\sin\theta\cos(\phi - \Omega\tau)}{l}\right)$$

$$\frac{1}{R} = \frac{1}{|\mathbf{x} - \mathbf{y}|} \approx \frac{1}{|\mathbf{x}|} = \frac{1}{l}$$

The pressure perturbation Eq. (24) can be rewritten in terms of the physical problem:

$$C_n = -\frac{ni\Omega^2}{8\pi^2 c_0}\int_0^{2\pi/\Omega}\left[\frac{D\sin\theta\sin(\phi - \Omega(\tau + R/c_0)) - T\cos\theta}{l}\right]e^{in\Omega(\tau + R/c_0)}d\tau \tag{25}$$

Recall that the coordinates system described in Figure 1 is placed arbitrary on plane of rotation, thus this degree of freedom can be harnessed into simplifying the problem. Assume that coordinate system is placed so that $\phi = \pi/2$ then Eq. (25) takes the form of:

$$C_n = -\frac{ni\Omega^2}{8\pi^2 c_0}\int_0^{2\pi/\Omega}\left[\frac{D\sin\theta\cos(\Omega(\tau + R/c_0)) - T\cos\theta}{l}\right]e^{in\Omega(\tau + R/c_0)}d\tau \tag{26}$$

Under the assumption that the system hold a periodic nature, the unsteady force fluctuations are as well periodic functions, therefor can be well represented by a Fourier series.

$$F = \sum_{\lambda=-\infty}^{\infty} F_\lambda e^{-i\lambda\Omega\tau} \tag{27}$$

where F_λ consist of thrust and drag components. In order to solve the integral in Eq. (26) the following transformation is suggested:

$$\begin{aligned} \xi &= \Omega\tau \\ \frac{d\xi}{d\tau} &= \Omega \end{aligned} \tag{28}$$

Apply Eqs. (27) & (28) to Eq. (26):

$$C_n = -\frac{ni\Omega}{8\pi^2 c_0}\int_0^{2\pi}\sum_{\lambda=-\infty}^{\infty}\left[\frac{D_\lambda \sin\theta\cos(\xi+\Omega R/c_0) - T_\lambda\cos\theta}{l}\right]\times \\ e^{\frac{in\Omega l}{c_0}} e^{i(n-\lambda)\xi - \frac{in\Omega r}{c_0}\sin\theta\sin\xi} d\xi \tag{29}$$

and assuming $\cos(\xi + \Omega R/c_0) \approx \cos\xi$. Furthermore, the main purpose in the development of the pressure perturbation term is to find its magnitude therefor the term $e^{\frac{in\Omega l}{c_0}}$ can be neglected, bounded by unity.

$$C_n = -\frac{ni\Omega}{8\pi^2 c_0}\int_0^{2\pi}\sum_{\lambda=-\infty}^{\infty}\left[\frac{D_\lambda \sin\theta\cos\xi - T_\lambda\cos\theta}{l}\right] e^{i(n-\lambda)\xi - \frac{in\Omega r}{c_0}\sin\theta\sin\xi} d\xi \tag{30}$$

Denote $M_\theta = \frac{\Omega r}{c_0}\sin\theta$:

$$C_n = -\frac{ni\Omega}{8\pi^2 c_0}\int_0^{2\pi}\sum_{\lambda=-\infty}^{\infty}\left[\frac{D_\lambda \sin\theta\cos\xi - T_\lambda\cos\theta}{l}\right] e^{i(n-\lambda)\xi - inM_\theta\sin\xi} d\xi \tag{31}$$

Assuming that the Fourier series converges, the integral and sum can be replaced. Thus the solution to the integral in Eq. (31) could be achieved by the integral definition of Bessel function of the first kind:

$$J_\nu(x) = \frac{1}{2\pi i^{-\nu}}\int_0^{2\pi} e^{i(\nu\eta - x\sin\eta)} d\eta \\ J_\nu(x) = -\frac{x}{\nu 2\pi i^{-\nu}}\int_0^{2\pi} e^{i(\nu\eta - x\sin\eta)}\cos\eta d\eta \tag{32}$$

So Eq. (31) in manners of Bessel functions is:

$$C_n = \frac{n\Omega}{4\pi l c_0}\sum_{\lambda=-\infty}^{\infty} i^{1-n+\lambda}\left[T_\lambda\cos\theta + \frac{(n-\lambda)}{nM_\theta}D_\lambda\sin\theta\right] J_{n-\lambda}(nM_\theta) \tag{33}$$

An important attribute of Bessel function of the first kind that the maximal amplitude of $J_\nu(x)$ decreases as ν increases, thus the terms containing $\lambda \leq 0$ can be neglected. Furthermore, applying Eq. (33) on a propeller multiple blades (B), the n-th Harmonic must be multiplied by the number of blades:

$$C_{mB} = \frac{mB^2\Omega}{4\pi l c_0}\sum_{\lambda=1}^{\infty} i^{1-(mB-\lambda)}\left[T_\lambda\cos\theta + \frac{(mB-\lambda)}{mBM_\theta}D_\lambda\sin\theta\right] J_{mB-\lambda}(mBM_\theta) \tag{34}$$

Recall that the aerodynamic force acting on the fluid equals to the spanwise force integral, hence:

$$F = \int_0^{d/2}\frac{2}{d}F(r)dr \tag{35}$$

where d is blade diameter. Therefor, Eq.(34) can be modified to fit this notion:

$$C_{mB} = \frac{mB^2\Omega}{4\pi l c_0} \sum_{\lambda=1}^{\infty} i^{1-(mB-\lambda)} \int_0^{d/2} \frac{2}{d} \left[T_\lambda(r)\cos\theta + \frac{c_0(mB-\lambda)}{mB\Omega r \sin\theta} D_\lambda(r)\sin\theta \right] \times J_{mB-\lambda}\left(\frac{mB\Omega r}{c_0} \sin\theta \right) dr \tag{36}$$

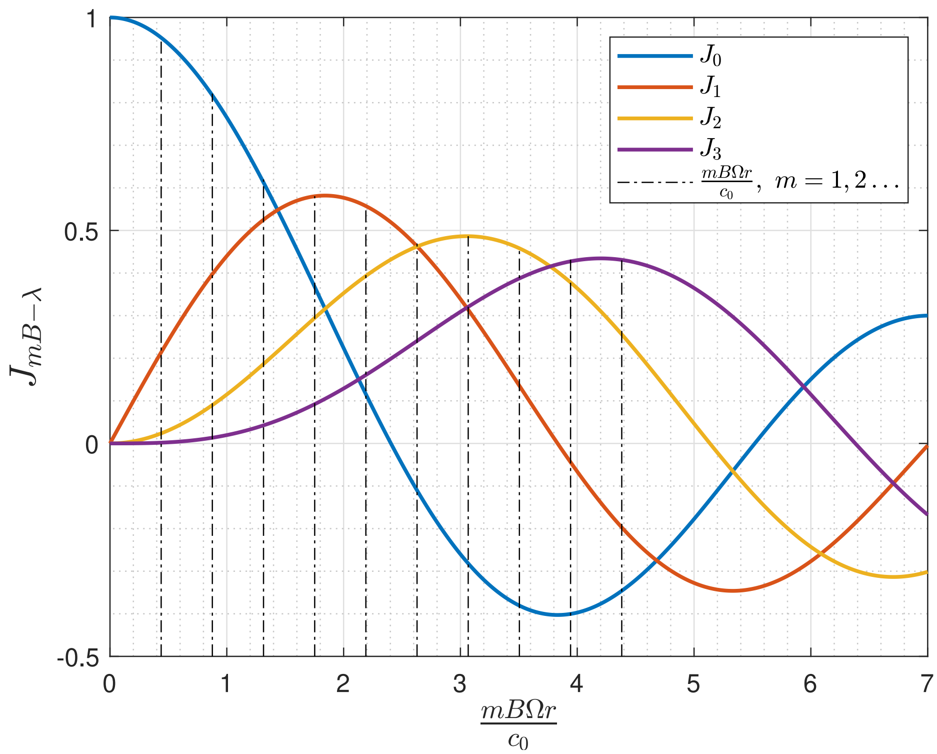


Figure 2: Relation between Besssel function of the 1st kind and n-th harmony of a two bladed 14-inch propeller at 4000 RPM.

In order to grasp the relationship between Bessel functions and the n-th harmonic pressure perturbation, a typical values regarding the physical system assumed and plotted as seen in Fig. 2. As depicted in Fig. 2, as the order of J_ν grows, i.e. ν grows, the magnitude of J_ν decrease. Thus it can be said that the lower harmonies are the dominant ones.

2.7 Loewy's Deficiency Function

In hover flight, the aerodynamic force acting on the fluid by the blade exhibits an unsteady behavior. One reason for this force fluctuations is due to returning wake effect. In order to model the effect of a returning wake generating the unsteady force, Loewy [2] suggested a deficiency function representing the magnitude of the mentioned force. Loewy [2] assumed a thin airfoil subjected to sinusoidal motion, thus the downwash velocity and its vorticity can be represented by

$$\begin{aligned} v_a &= \bar{v}_a e^{i\omega t} \\ \gamma_a &= \bar{\gamma}_a e^{i\omega t} \end{aligned} \tag{37}$$

Assuming only the vorticity within the proximity of the blade has significant contribution, the rows of vortex sheets may be allowed to extend to infinty in order to simplify the problem mathematically.

Thus, according to the Biot-Savart theorem, the induced velocity at some point x' is:

$$dv_a = \frac{\gamma_{nq}(x' - \xi')d\xi}{2\pi\left[(x' - \xi')^2 + (nB + q)^2(h')^2\right]} \tag{38}$$

where γ_{nq} is the vortex sheet shed by q-th blade in the n-th revolution, B is the number of blades and h' is the vertical distance between successive rows of sheets. Loewy's model is depicted in Fig. 3.

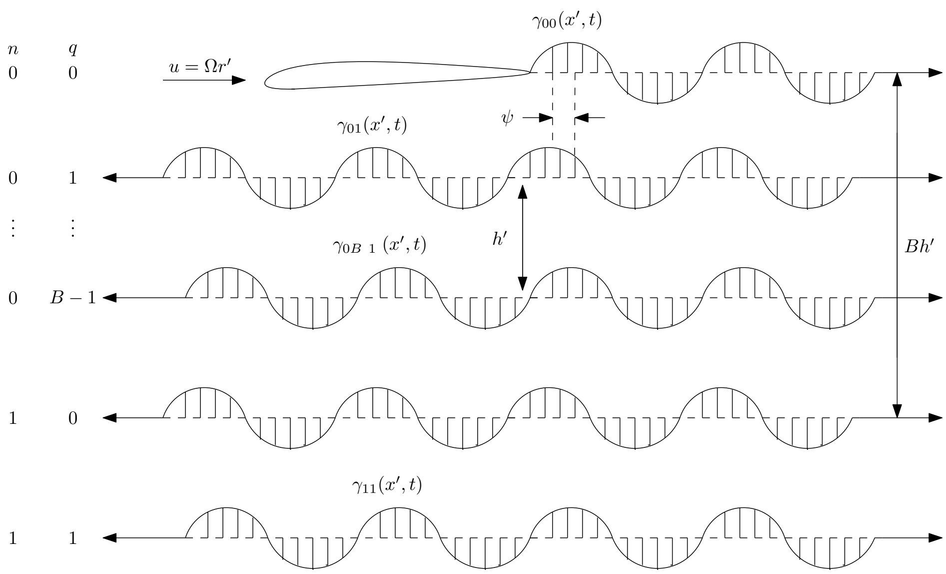


Figure 3: Lowey's aerodynamic model of a multi-blade propeller.

The downwash can be calculated as the sum of all first-order effects of wake, i.e. the effects on wake on itself is neglected, therefor:

$$v_a(x',t) = -\frac{1}{2\pi}\Bigg[\int_{-b}^{b}\frac{\gamma_a(\xi',t)d\xi}{x'-\xi'} + \int_{b}^{\infty}\frac{\gamma_{00}(\xi',t)d\xi'}{x'-\xi'} + \sum_{q=1}^{B-1}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{\gamma_{nq}(\xi',t)(x'-\xi')d\xi'}{(x'-\xi')^2+(nB+q)^2(h')^2} + \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{\gamma_{n0}(\xi',t)(x'-\xi')d\xi'}{(x'-\xi')^2+n^2B^2(h')^2}\Bigg] \quad (39)$$

where the 1st term on the RHS represent the effect of the reference blade, 2nd term the effect of the attached shed vortex sheet, 3rd term the effect of both attached and separated vortex sheets created by other blades and 4th term the effect of separated sheets from the reference blade.

Recall the sinusoidal motion assumption in Eq. (37), thus the vorticity shed by the qth blade at the n-th rotation with respect to the reference rotation can be presented by:

$$\gamma_{nq} = \bar{\gamma}_{0q}e^{i\omega[t-(2\pi n/\Omega)]} = \bar{\gamma}_{0q}e^{-i2\pi n(\omega/\Omega)}e^{i\omega t} \quad (40)$$

thus:

$$\bar{\gamma}_{nq} = \bar{\gamma}_{0q}e^{-i2\pi(\omega/\Omega)n} \quad (41)$$

Define the following quantities:

$$m \triangleq \omega/\Omega; \quad k \triangleq (\omega/\Omega)(c/2r') = m(c/2r'); \quad \bar{\Gamma} \triangleq (2\bar{\Gamma}_0'/c)e^{ik}$$

where m is the ratio between the oscillatory frequency and the angular velocity of the blade, that helps defining the reduced frequency k which gives an indication for the unsteadiness of the system. Typically a flow is considered unsteady when $k > 0.05$. Including the definitions above and the appearance of a phase between shed vortex sheets:

$$\bar{\gamma}_{nq} = -ik\bar{\Gamma}e^{i\psi_q}e^{-i(m\xi'/r')}e^{-i2\pi(q/B)m}e^{-i2\pi nm} \quad (42)$$

Substituting Eqs. (37) & (42) into the downwash Eq. (39):

$$\bar{v}_a(x') = -\frac{1}{2\pi}\Bigg[\int_{-b}^{b}\frac{\bar{\gamma}_a(\xi')d\xi'}{x'-\xi'} - ik\bar{\Gamma}\int_{b}^{\infty}\frac{e^{-i(m\xi'/r')}d\xi'}{x'-\xi'} - ik\bar{\Gamma}\sum_{q=1}^{B-1}\sum_{n=0}^{\infty}e^{-i\{2\pi m[n+(q/B)]-\psi_q\}}\int_{-\infty}^{\infty}\frac{e^{-i(m\xi'/r')}(x'-\xi')d\xi'}{(x'-\xi')^2+(nB+q)^2(h')^2} - ik\bar{\Gamma}\sum_{n=1}^{\infty}e^{-i2\pi mn}\int_{-\infty}^{\infty}\frac{e^{-i(m\xi'/r')}(x'-\xi')d\xi'}{(x'-\xi')^2+n^2B^2(h')^2}\Bigg] \quad (43)$$

Divide Eq. (43) by semi-chord b in order to non-dimensionalize it:

$$\bar{v}_a(x) = -\frac{1}{2\pi}\left[\int_{-1}^{1}\frac{\bar{\gamma}_a(\xi)d\xi}{x-\xi} - ik\bar{\Gamma}\int_{1}^{\infty}\frac{e^{-ik\xi}d\xi}{x-\xi} - ik\bar{\Gamma}\sum_{q=1}^{B-1}e^{i[\psi_q-2\pi m(q/B)]}\sum_{n=0}^{\infty}e^{-i2\pi mn}\int_{-\infty}^{\infty}\frac{e^{-ik\xi}(x-\xi)d\xi}{(x-\xi)^2+(nB+q)^2(h)^2} - ik\bar{\Gamma}\sum_{n=1}^{\infty}e^{-i2\pi mn}\int_{-\infty}^{\infty}\frac{e^{-ik\xi}(x-\xi)d\xi}{(x-\xi)^2+n^2B^2h^2}\right] \tag{44}$$

Notice the last two terms on the RHS, both include an integrals of the form:

$$\int_{-\infty}^{\infty}\frac{e^{-ik\xi}(x-\xi)d\xi}{(x-\xi)^2+A^2}$$

by a simple transformation $x-\xi=-A\lambda$ the integral above can be solved:

$$\int_{-\infty}^{\infty}\frac{e^{-ik\xi}(x-\xi)d\xi}{(x-\xi)^2+A^2} = i\pi e^{-k(ix-A)} \tag{45}$$

Furthermore, the terms containing summation over n are cornvergent geometric series (kBh always positive). Substituting the integral solution Eq. (45) to the mentioned terms and using the converging geometric series yields:

$$-ik\bar{\Gamma}\sum_{q=1}^{B-1}e^{i[\psi_q-2\pi m(q/B)]}\sum_{n=0}^{\infty}e^{-i2\pi mn}\int_{-\infty}^{\infty}\frac{e^{-ik\xi}(x-\xi)d\xi}{(x-\xi)^2+(nB+q)^2(h)^2} = k\bar{\Gamma}\pi e^{-ikx}\frac{\sum_{q=1}^{B-1}e^{i\psi_q-(q/B)[2\pi mi+kBh]}}{1-e^{2\pi mi+kBh}} \tag{46}$$

$$-ik\bar{\Gamma}\sum_{n=1}^{\infty}e^{-i2\pi mn}\int_{-\infty}^{\infty}\frac{e^{-ik\xi}(x-\xi)d\xi}{(x-\xi)^2+n^2B^2h^2} = k\bar{\Gamma}\pi e^{-ikx}\frac{e^{2\pi mi+kBh}}{1-e^{2\pi mi+kBh}} \tag{47}$$

So Eq. (45) can be rewritten as:

$$\bar{v}_a = -\frac{1}{2\pi}\left[\int_{-1}^{1}\frac{\bar{\gamma}_a(\xi)d\xi}{x-\xi} - ik\bar{\Gamma}\int_{1}^{\infty}\frac{e^{-ik\xi}d\xi}{x-\xi} + \pi k\bar{\Gamma}e^{-ikx}W\right] \tag{48}$$

where

$$W = W(kh, m, B, \psi_q) = \frac{1+\sum_{q=1}^{B-1}\left(e^{kBh}e^{i2\pi m}\right)^{(B-q)/B}e^{i\psi_q}}{e^{kBh}e^{i2\pi m}-1} \tag{49}$$

In order to calculate W, the phase angle between successive vortex sheets is needed to be approximated. Since the vortex is shedding frequency is ω and the phase occurs between two successive sheets, i.e. between a vortex sheet shed at previous rotation of the reference blade, the phase angle $\psi_q \sim 2\pi\omega/\Omega$. Furthermore, in the general case of a multibladed propeller, the phase angle must be related to the blade index, thus $\psi_q = 2\pi(q/B)(\omega/\Omega)$.

Recall that W was derived from a mathematical simplification of the terms related to effect of vortex shed by preceding blades and previous revolutions of the reference blade, so W can be treated as a weighting function for these effects.

Keeping in mind that in order to calculate the pressure perturbation, the vorticity is needed, i.e. the inverse problem to the one presenten in Eq. (48). In order to do so, Söhngen's [3] derivation is used, as presented by Loewy [2]. Assume an expression of the form:

$$g(x) = \frac{1}{2\pi}\int_{-1}^{1}\frac{f(\xi)d\xi}{x-\xi} \tag{50}$$

so according to Söhngen [3], the inverse problem is:

$$f(x) = -\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{g(\xi)d\xi}{(x-\xi)} \tag{51}$$

if $f(1)$ is finite. In order to apply the inverse problem form presented above to Eq. (48), $\bar{\gamma}_a(1)$ must be finite, the Kutta condition must be employed, therefor:

$$\begin{aligned}\bar{\gamma}_a(x) = \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\Bigg[\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{\bar{v}_a(\xi)d\xi}{(x-\xi)} - \\ \frac{ik\bar{\Gamma}}{2\pi}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{1}{(x-\xi)}\int_{1}^{\infty}\frac{e^{ik\eta}d\eta}{\xi-\eta}d\xi + \frac{k\bar{\Gamma}}{2}W\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{e^{ik\xi}d\xi}{(x-\xi)}\Bigg]\end{aligned} \tag{52}$$

Evaluating the circulation over the airfoil $\bar{\Gamma}$ via

$$\bar{\Gamma} = e^{ik}\int_{-1}^{1}\bar{\gamma}_a(x)dx$$

yields:

$$\bar{\Gamma} = \frac{2\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\bar{v}_a(\xi)d\xi}{i\pi k\left[\frac{1}{2}(H_1^{(2)}(k) + iH_0^{(2)}(k)) + W(J_1(k) + iJ_0(k))\right]} \tag{53}$$

where $H_\nu^{(2)}(k)$ is Henkle function.

Bernnoulli's unsteady equation in non-dimentional length terms:

$$\frac{\partial\phi}{\partial t} + \frac{2V}{c}\frac{\partial\phi}{\partial x} + \frac{p}{\rho} = f(t) \tag{54}$$

where ϕ is velocity potential. Recall that according Loewy model assumes a thin airfoil, i.e. the airfoil is replaced by a vortex sheet, therefor the difference between the velocity above and below the airfoil is equal to $\gamma_a(x,t)$, so

$$\frac{\partial\phi_U}{\partial x} - \frac{\partial\phi_L}{\partial x} = \frac{c}{2}\gamma_a(x,t)$$
$$\phi_U - \phi_L = \frac{c}{2}\int_{-1}^{x}\gamma_a(\xi,t)d\xi$$

Apply the terms above to Eq. (54):

$$\Delta p \triangleq p_U - p_L = -\rho \left[V\gamma_a(x,t) + \frac{c}{2}\frac{\partial}{\partial t}\int_{-1}^{x} \gamma_a(\xi,t)d\xi \right] \tag{55}$$

and for sinusoidal oscillatory motion, as depicted in Eq. (37), Bernnoulli's unsteady equation in non-dimentional length terms takes form:

$$\frac{\Delta\bar{p}(x^*)}{\rho V} = -\bar{\gamma}_a(x^*) - ik\int_{-1}^{x^*} \gamma_a(\xi^*)d\xi^* \tag{56}$$

Applying Eqs. (52)-(53) to the equation above, yields

$$\begin{aligned} \frac{-\Delta\bar{p}(x^*)}{\rho r\Omega} = & \frac{(2i/\pi)[H_0^{(2)}(k) + 2J_0(k)W]}{H_1^{(2)}(k) + iH_0^{(2)}(k) + W[J_1(k) + iJ_0(k)]} \int_{-1}^{1} \sqrt{\frac{1-x^*}{1+x^*}}\sqrt{\frac{1+\xi}{1-\xi}}\bar{v}_a(\xi)d\xi + \\ & \frac{2}{\pi}\int_{-1}^{1}\left[\sqrt{\frac{1-x^*}{1+x^*}}\sqrt{\frac{1+\xi}{1-\xi}}\frac{1}{x^*-\xi} - ik\varphi(x^*,\xi)\right]\bar{v}_a(\xi)d\xi \end{aligned} \tag{57}$$

where

$$\varphi(x^*,\xi) = \frac{1}{2}\ln\left[\frac{1-x^*\xi + \sqrt{1-\xi^2}\sqrt{1-x^{*2}}}{1-x^*\xi - \sqrt{1-\xi^2}\sqrt{1-x^{*2}}}\right] \tag{58}$$

Eq. (57) is with the same form of the two-dimensional fixed-wing oscillating in incompressible flow derived by Theodorsen [4]. Denote the multiplier of the first term on RHS of Eq. (57):

$$\frac{2}{\pi}[1 - C'(k,m,h)] = \frac{(2i/\pi)[H_0^{(2)}(k) + 2J_0(k)W]}{H_1^{(2)}(k) + iH_0^{(2)}(k) + W[J_1(k) + iJ_0(k)]} \tag{59}$$

so

$$C'(k,m,h) = F' + iG' = \frac{H_1^{(2)}(k) + 2J_0(k)W(kh,m)}{H_1^{(2)}(k) + iH_0^{(2)}(k) + 2[J_1(k) + iJ_0(k)]W(kh,m)} \tag{60}$$

where $h = \frac{4\pi v_i}{cB\Omega}$, $k = \frac{c\omega}{2V}$ and $\omega = \Omega mB$. The function $C'(k,m,h)$ called Loewy's Deficiency Function (LDF), and due to the similar form of Eq. (57) to Theodorsen's model it is analogues to Theodorsen's circulatory lift function $C(k)$.

2.8 Unsteady Lift Calculation

When calculating the unsteady lift via LDF, the fact that C' is complex is needed to be taken into account. Recall that C' was derived as a force response to an harmonic downwash velocity input, thus C' represent both amplitude and phase between the input downwash velocity and the output force. In aspect of blade noise, only that amplitude matters. Starting with lift on a thin airfoil:

$$L = \frac{1}{2}\rho V^2 bc(2\pi\alpha) \tag{61}$$

Assuming $v_i \ll V$ thus $\alpha \approx \frac{v_i}{V}$ so:

$$L = \pi\rho V v_i bc = \pi\rho V \bar{v}_i bce^{i\omega t} \tag{62}$$

Implementing LDF in order to achieve the unsteady force amplitude:

$$|L_{us}| = \pi\rho V \bar{v}_i bc|C'| \tag{63}$$

where $\overline{v}_i$ can be calculated via blade element momentum theory (BEMT) or an aero-prediction tool such as XFOIL [5]. Applying the unsteady lift term to the n-th harmonic pressure perturbation Eq. (36) where the quasi-steady term, i.e. the nominal thrust and drag, is the first term in Fourier series:

$$C_N = \frac{mB^2\Omega}{4\pi l c_0}\left[i^{2-mB}\int_0^{d/2}\frac{2}{d}\left(T_{\text{nom}}\cos\theta + \frac{c_0(mB-1)}{mB\Omega r}D_{\text{nom}}\right)J_{mB-1}\left(\frac{mB\Omega r}{c_0}\sin\theta\right)dr + \sum_{\lambda=2}^{\infty} i^{1-(mB-\lambda)}\int_0^{d/2}\frac{2}{d}\left(|L_{us}(k,N,h)|\cos\theta + \frac{c_0(mB-\lambda)}{mB\Omega r}D_\lambda(r)\right)\times J_{mB-\lambda}\left(\frac{mB\Omega r}{c_0}\sin\theta\right)dr\right] \tag{64}$$

3 Acoustic Analysis of Typical Parameters

Table 1 lists the typical parameters for a two bladed propeller.

c	B	Ω	N	r	T	D	v_i	V	h	$\lvert C'\rvert$
0.02	2	66.67	1	0.1334	9.5	0.95	5	62	3.3	1.0328
0.02	2	66.67	2	0.1334	9.5	0.95	5	62	3.3	0.6433
0.02	2	66.67	3	0.1334	9.5	0.95	5	62	3.3	0.9272
0.02	2	66.67	4	0.1334	9.5	0.95	5	62	3.3	0.6040
0.02	2	66.67	5	0.1334	9.5	0.95	5	62	3.3	0.8104

Table 1: Typical parameters.

3.1 Spanwise Force & Chord Distribution Approximation

In order to approximate the tonal noise components, the properties of the blade is needed to be known. Assuming a spanwise force distribution function:

$$F(x) = F_{\text{nominal}}\frac{1}{A}\left[\frac{x}{r} - \left(\frac{x}{r}\right)^2\right]e^{B\frac{x}{r}}$$
$$A = 1.385448019 \tag{65}$$
$$B = \frac{8}{3}$$

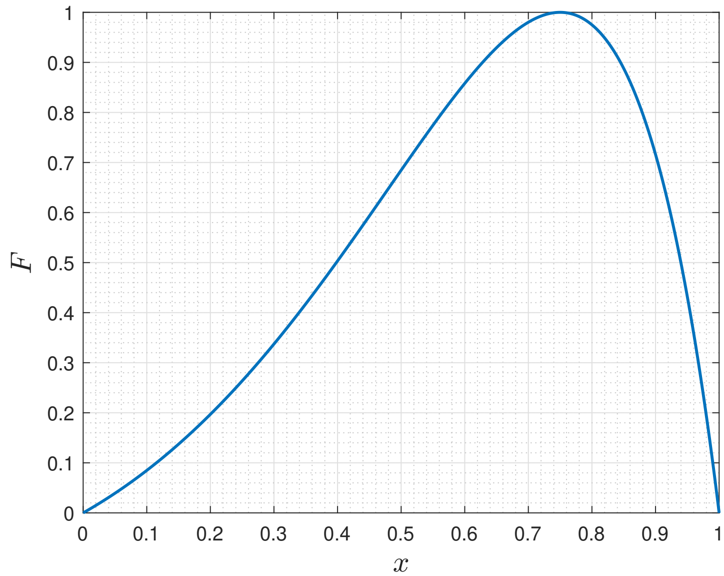


Figure 4: Force spanwise distribution function.

The values of A, B were determined so that $F(x)$ equals to F_{nominal} at 75% span. Furthermore, the drag term is approximated as 10% of the nominal thrust.

Beside the aerodynamic force, the spanwise chord is needed to be approximated as well. Assume a spanwise chord distribution function:

$$\begin{aligned} c(x) &= c_{\max} A \left(\frac{x}{r} - 1\right) \sin\left(2\frac{x}{r} + B\right) \\ A &= \frac{-4\sqrt{13}}{9} \\ B &= -0.5 + 2\arctan\left(\frac{\sqrt{13}-2}{3}\right) \end{aligned} \tag{66}$$

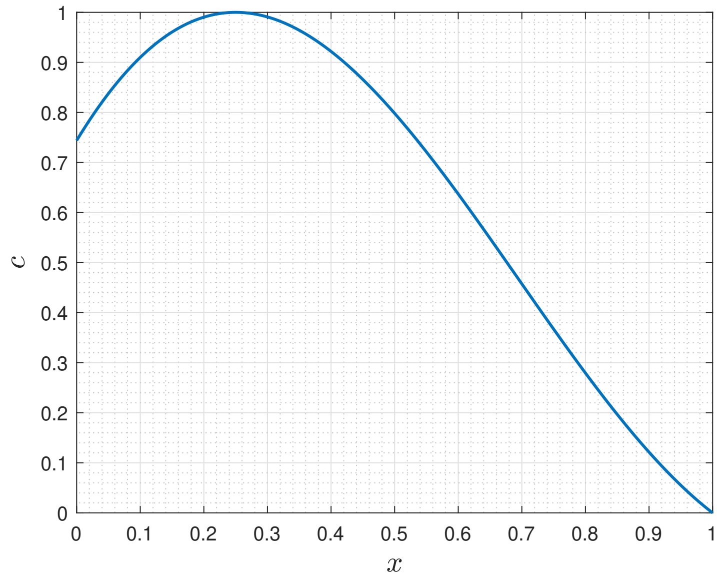
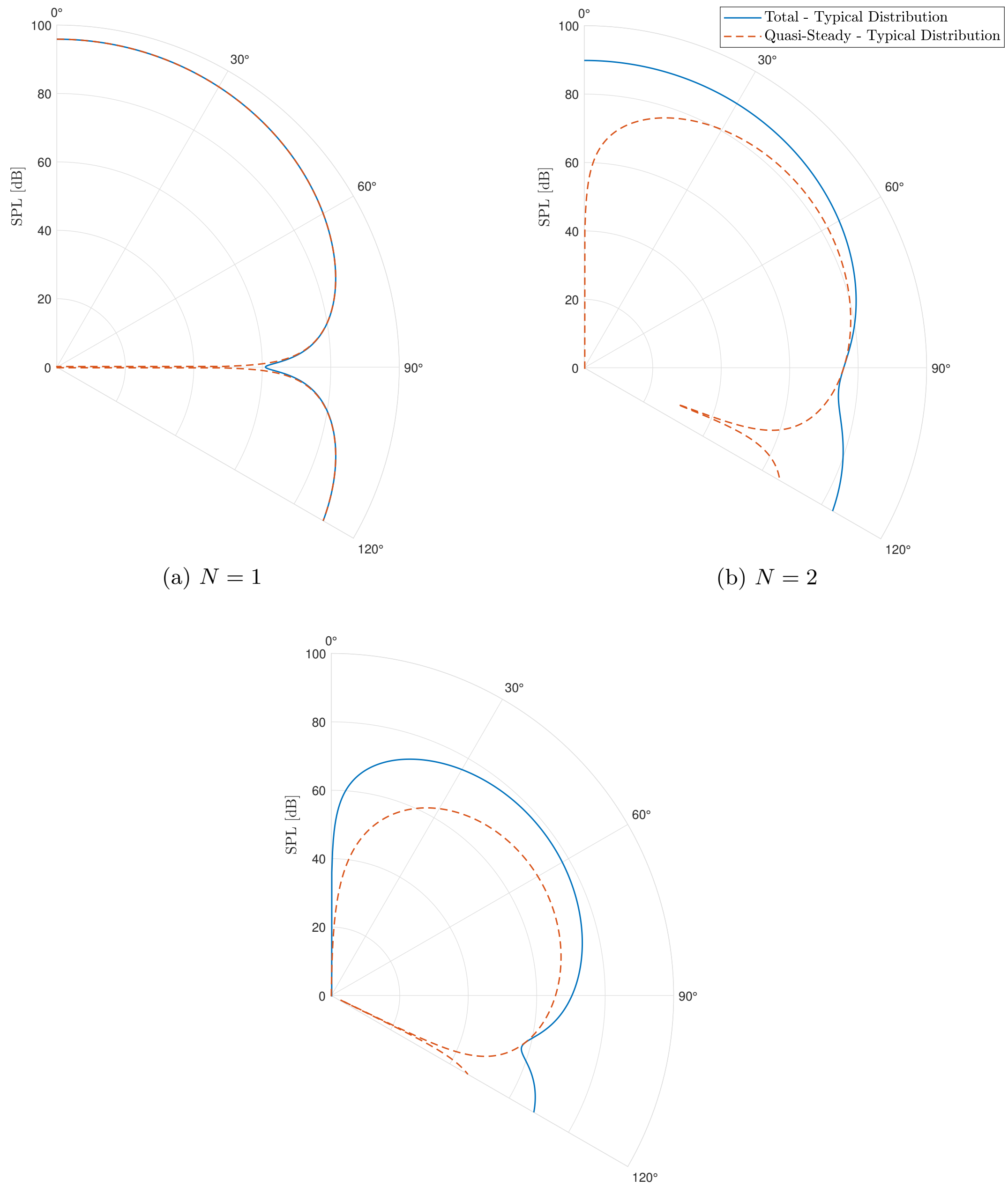


Figure 5: Chord spanwise distribution function.

where the values of A and B were determined so that $c(x)$ equals to $c_{\max}$ at 25% span.

3.2 Acoustic Analysis - Spanwise Distribution Approximation

Implementing to Eq. (64), force and chord spanwise distributions Eqs. (65)-(66) and solving numerically. The first three harmonics are presented in Fig. 6 for both unsteady and quasi-steady cases for the typical parameters presented in Table 1, where $\Omega = 66.67$ at a distance of 1.5 [m] from the hub.



(a) $N = 1$

(b) $N = 2$

(c) $N = 3$

Figure 6: Pressure perturbation harmonics - typical force distribution.

As seen in Fig. 6, the effect of the unsteady force to the pressure perturbation, is significant in comparison to quasi-steady force, especially in the second harmony. An approximation of the acoustic signature of a typical blade represented by the sum of the first 3 pressure perturbation harmonics is depicted in Fig. 7.

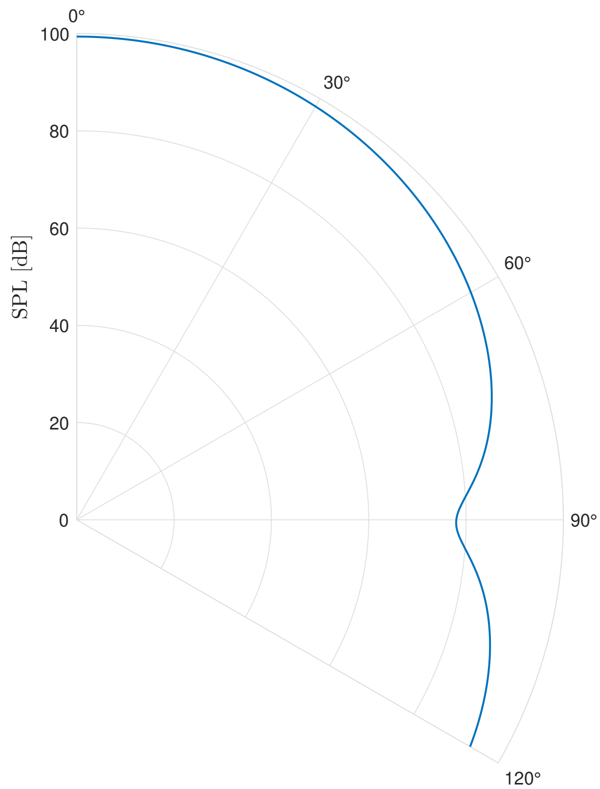


Figure 7: Approximation of the acoustic signature of a typical blade - typical force distribution.

3.3 Acoustic Analysis - BEMT Data

In order to validate the assumed force and spanwise chord distributions Eqs. (65)-(66) a values calculated via Blade Element Momentum Theory (BEMT), presented in Fig. 8, are taken and compared to the pressure perturbations calculated by the mentioned distribution functions.

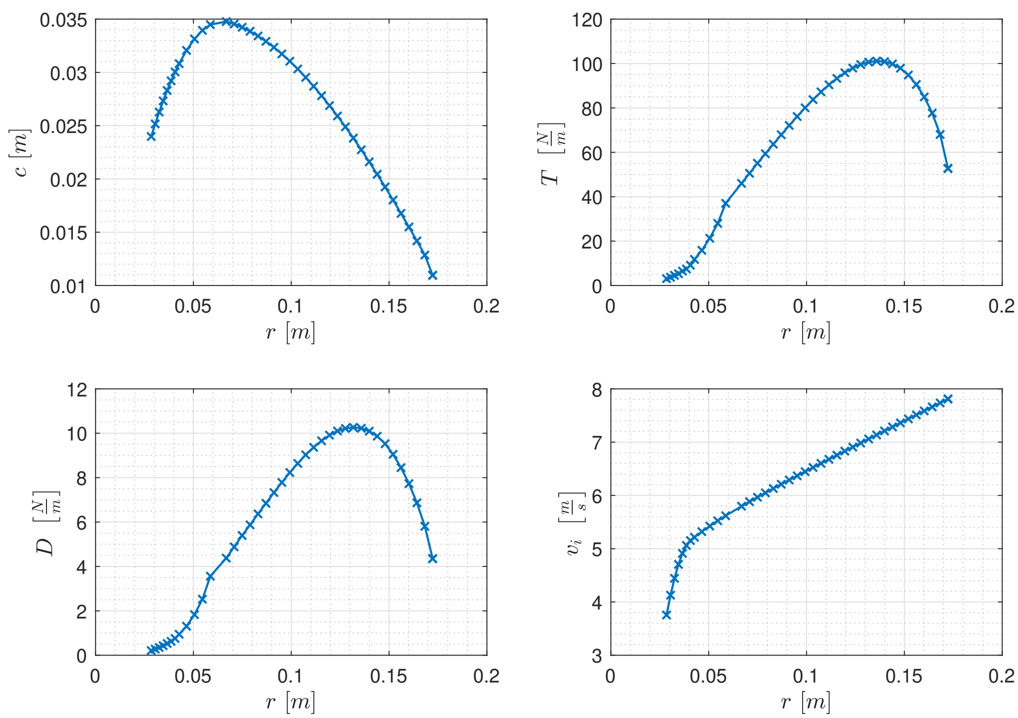
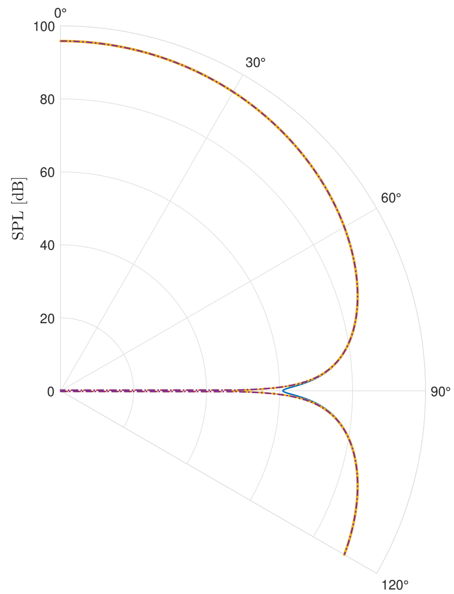
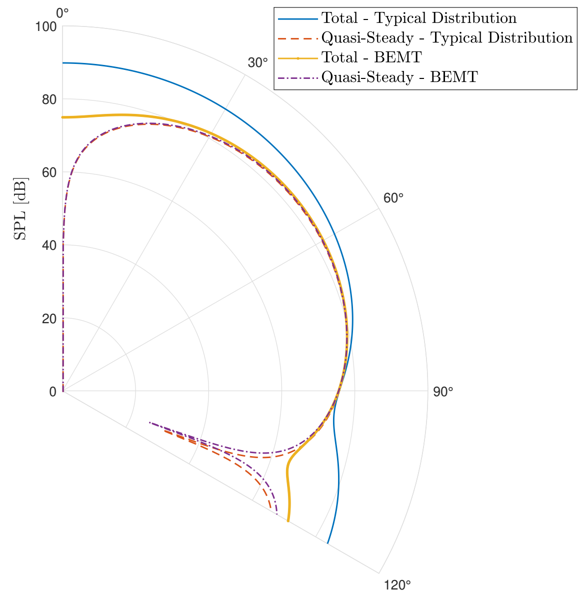


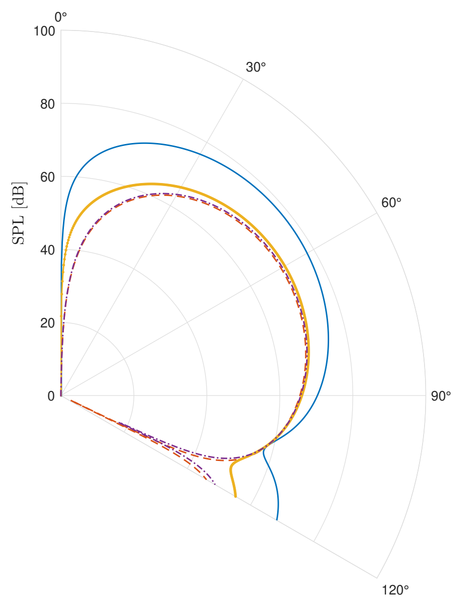
Figure 8: BEMT parameters.



(a) $N = 1$



(b) $N = 2$



(c) $N = 3$

Figure 9: Pressure perturbation harmonics - typical force distribution versus BEMT force distribution.

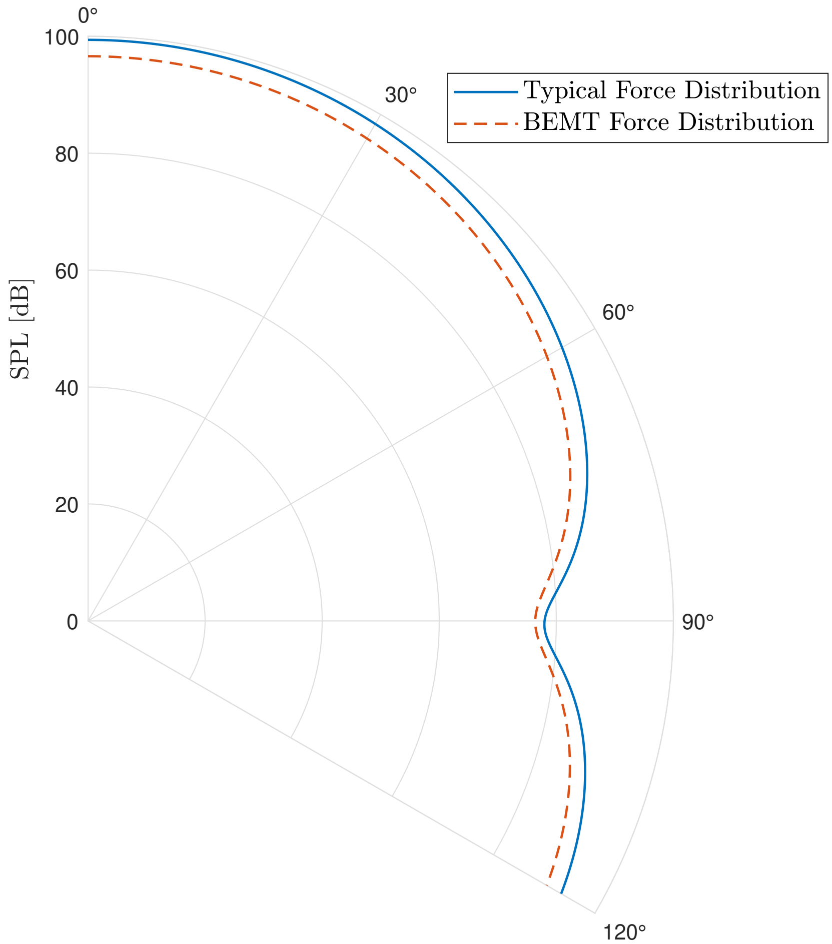


Figure 10: Approximation of the acoustic signature of a typical blade - typical force distribution versus BEMT force distribution.

From Figs. 9-10 it can be concluded that the force and spanwise chord distributions approximated functions (65) & (66) yields an adequate approximation to the acoustic signature of a rotating blade.

Furthermore, a comparison between the approximated force distribution function for thrust and drag and the BEMT is presented in Fig. 11.

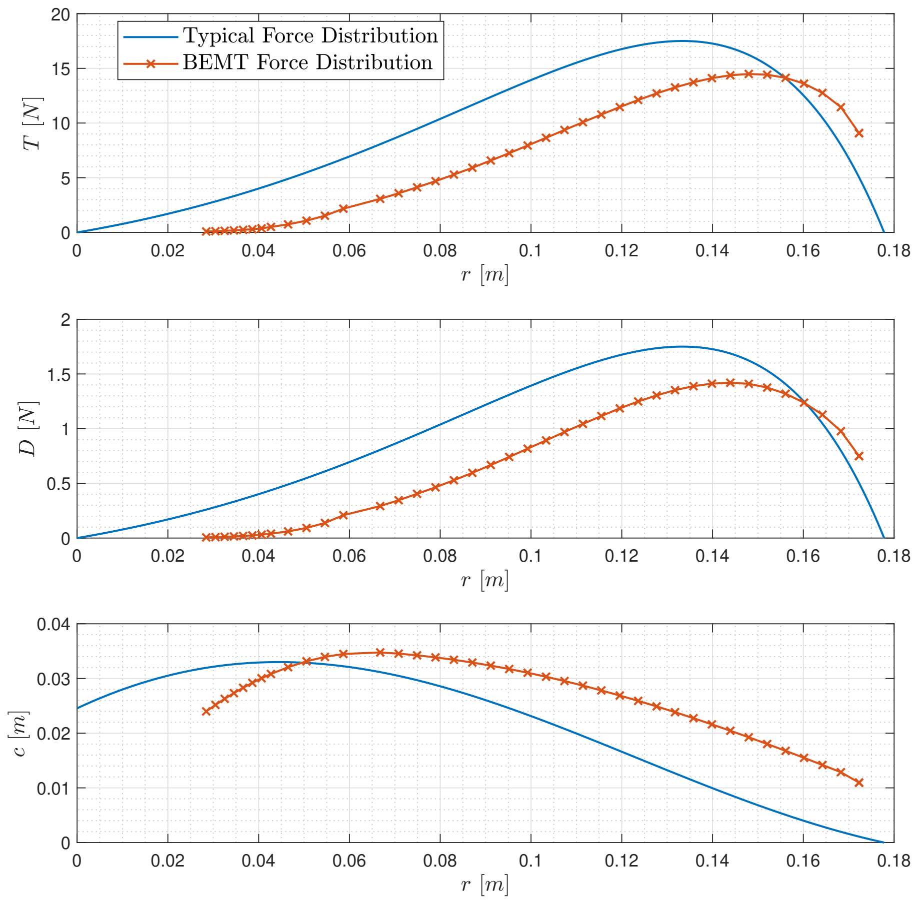


Figure 11: Approximated force & chord distribution versus BEMT force chord distribution.

4 Conclusions

From the analysis presented above, two main conclusions emerge:

1. The unsteady force generated by wake related effects described by Loewy [2] contribute significantly to high order pressure perturbation harmonics, i.e. 2-nd harmony and above. Moreover, the pressure harmonics calculated using LDF captures perturbations at certain θ angles that the quasi-steady case seem to omit.

2. For the quasi-steady case, The effect of spanwise force distribution on seem to have a minor effect. On the other hand when taking into account the unsteady force effect on the pressure perturbation harmonics, the chord spanwise distribution and the induced velocity are dominant, due to the dominant contribution of the unsteady lift component Eq. 63. Thus the spanwise chord distribution can be used as an acoustic design parameters, even tough the spanwise force distribution is strongly related to the chord spanwise distribution for a given nominal quasi-steady force generated by a blade an acoustically optimal chord spanwise distribution can be found.

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