Faculty of Aerospace Engineering – Technion - Israel Institute of Technology

Research Project

Transition to Turbulence in a Pipe – Part A – Simulation of Algebraic Growth of Perturbations

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Abstract

This research project deals with direct numerical simulation (DNS) of transition from laminar to turbulent flow in pipe Poiseuille flow. The transition is modeled by a combination of a primary disturbance, undergoing transient growth in time, and a secondary disturbance, growing exponentially. This report focuses on the transient growth of the primary disturbance. The numerical results are compared to an analytical model based on a four-mode approximation for various initial amplitudes. A follow-up project concerns the evolution of the secondary disturbance and the transition to turbulence.

I. Introduction

The purpose of this project is to try to predict transition from laminar flow to turbulence in pipe Poiseuille flow. According to the linear stability theory (LST) the flow is linearly stable, although it has been shown in experiments that at Reynolds numbers (based on the axial velocity and half of the pipe diameter) greater than $Re \approx 2000$ transition can occur. For a viscous base flow, it has been shown that a disturbance can undergo appreciable initial growth before eventually decaying due to viscous effects. This mechanism is called transient growth (TG). Therefore, a small disturbance added to the base flow can be significantly amplified by the TG mechanism, such that the modified flow may become unstable to secondary disturbances. Studies of optimal initial disturbances have shown that the maximal growth is obtained by a streamwise independent disturbance consisting of a counter-rotating vortex pair (CVP), whose velocity field is schematically in Figure [3]. Previous studies of subcritical transition in Couette flow [1] and plane Poiseuille flow [2] have shown that transient growth can be followed analytically using only the first four decaying modes. This report (part A) handles with simulation of algebraic growth of perturbations which means adding a main (primary) disturbance as a linear combination of four modes, while the follow-up report (part B) describes the addition of an infinitesimal three-dimensional secondary disturbance that may lead to transition to turbulence.

II. Mathematical Method

A. Direct Numerical Simulation

The computation of the transition scenario is done using a well-tested open source code written by Willis called 'Openpipeflow' DNS software [3]. The pipe has an axial (streamwise) coordinate x and its radius is $r = 1$. The length of the pipe is $L_x = 2\pi/\alpha$ where α is the axial (streamwise) wavenumber, so that the cylindrical coordinates are $(r, \theta, x) \in [0, 1] \times [0, 2\pi] \times [0, 2\pi/\alpha]$. The coordinate system as well as the geometry of the problem are illustrated in Figure [1]. The spatial discretisation of the code is double-Fourier (θ, x) and finite difference (r), distributed in a form of Chebyshev polynomial, such that points are clustered towards the boundaries. In addition, there is no point on the axis, $r = 0$, to avoid singularities in $1/r$ terms. The temporal discretisation is a second-order Predictor-Corrector scheme, with automatic time-step control. The code may be run either on a single core or in parallel (with MPI).

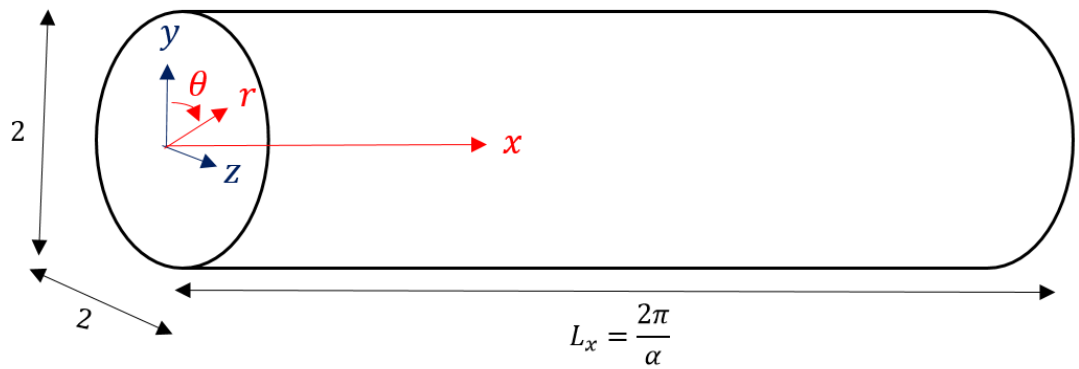


Figure 1: The geometry of the problem

B. Stability Analysis

The unperturbed velocity profile (called also the base flow) in pipe Poiseuille flow has a normalized form of $\boldsymbol{U}_0(r) = (0, 0, U_0(r))$, where $U_0(r) = 1 - r^2$. As usual in a linear stability analysis, we assume an infinitesimal disturbance $\boldsymbol{u}'$, i.e.

$$|\boldsymbol{u}'/\boldsymbol{U}_0| \ll 1, \tag{1}$$

such that the perturbed flow becomes

$$\boldsymbol{u}(t, r, \theta, x) = (u_r, u_\theta, u_x) = \boldsymbol{U}_0 + \boldsymbol{u}'. \tag{2}$$

Since the base flow does not depend on t, x, θ, a normal mode solution is sought

$$\boldsymbol{q}'(t, r, \theta, x) = \hat{\boldsymbol{q}}(r) e^{i(\alpha x + n\theta - \omega t)} = \hat{\boldsymbol{q}}(r) e^{i(\alpha x + n\theta - \omega_r t)} e^{\omega_i t}, \tag{3}$$

where $\boldsymbol{q} = (u_r; u_\theta; u_x; p)$, $\hat{\boldsymbol{q}}$ are complex functions, (t, r, θ, x) are the dimensionless time, radial, angular and axial coordinates respectively, α is the axial wavenumber ($\alpha \in \mathbb{R}$), n is the angular wavenumber ($n \in \mathbb{Z}$), and ω is the complex eigenvalue $\omega \in \mathbb{C}$ $(\omega = \omega_r + i\omega_i)$, where ω_r is the frequency and ω_i is the growth/decay rate. This case, where $\omega \in \mathbb{C}$, is termed the temporal problem. From equation (3) we can see that $\boldsymbol{q}' \propto \exp(\omega_i t)$, hence for $\omega_i > 0$ the disturbance grows, whereas for $\omega_i < 0$ it decays, and in the special case of $\omega_i = 0$ it neither grows nor decays (neutral). Using the normal mode ansatz, the linearized Navier–Stokes equations can be reduced to two equations for the radial velocity disturbance and the radial vorticity – The Orr–Sommerfeld (OS) and the Squire (Sq) equations. As the latter equation has only decaying solutions for the temporal case, the only possible source of exponential instability are the OS modes.

C. Analytical Model for Transient Growth

Transient growth is a possible mechanism for obtaining transition below the critical Reynolds number. In this mechanism, initial disturbance growth can be obtained by using a combination of linearly stable (decaying) modes, before ultimately decaying due to viscous effects. The goal is to find such an initial combination of modes that will lead to maximal growth of the initial disturbance. Transition to turbulence may be possible if during the transient growth, the modified base flow, consisting of the base flow and the disturbance, becomes unstable to secondary instabilities. Since the solutions of the OS and Sq equations are non-orthogonal, appreciable transient growth is possible. It turns out that the optimal disturbance is obtained for the streamwise-independent case, i.e. $\alpha = 0$. The reason transient growth is also called algebraic growth is because for inviscid flow by assuming an x-independent disturbance ($\partial/\partial x = 0$) one may get from the inviscid three-dimensional stability equations that the disturbance grows linearly with time, rather than exponentially. Studies have also shown that the maximal transient growth is obtained for $n = 1$. Karp, Roizner, and Cohen [4] showed for Couette flow and for plane Poiseuille flow that by considering only a small number of the least stable modes one can analytically predict the TG process resulting from many modes adequately. Comparison of the analytical model and DNS was performed for Couette flow [5] and plane Poiseuille flow [6, 2]. From the streamwise-independent stability analysis we know that all the modes are located on the vertical axis, when plotting ω_i as a function of ω_r (see, e.g. Figure [2] in Karp and Cohen [1] for Couette flow). Similarly to the previous work in Couette flow and plane Poiseuille flow, we will consider only the first 4 modes in the stability analysis and compare this approximation to the DNS results in order to evaluate the TG. The chosen Reynolds number for the analysis is $Re = 3000$, which is relatively low.

The TG based on 4 modes is given by

$$\boldsymbol{u}_{TG}(t,r,\theta) = \boldsymbol{U}_0(r) + \varepsilon \boldsymbol{u}_1(t,r,\theta) + \varepsilon^2 \boldsymbol{u}_2(t,r,\theta) + \cdots , \tag{4}$$

where ε is small ($\varepsilon \ll 1$), $\boldsymbol{U}_0$ is pipe Poiseuille flow, $\varepsilon \boldsymbol{u}_1$ is the velocity of the TG due to the 4 modes and it is of the order $\mathcal{O}(\varepsilon)$, and $\varepsilon^2 \boldsymbol{u}_2$ represents nonlinear interactions and this term is of the order $\mathcal{O}(\varepsilon^2)$. This nonlinear term can be solved analytically using Duhamel's principle.

This report discusses the optimal conditions for TG, validation of the solver and comparison of the DNS results with the four-mode approximations. Part B of this work addresses the addition an infinitesimal secondary disturbance, $\delta u_d(t,r,\theta,x)$ (order of a small parameter δ), to the modified base flow $\boldsymbol{u}_{TG}$, in order to obtain instability and transition to turbulence.

The linear term of the 4 modes, for the case of $\alpha = 0$, $n = 1$, is given by

$$\boldsymbol{u}_1(t,r,\theta) = A_0 \, \Re\left\{ \sum_{m=1}^{4} A_m \boldsymbol{u}_m(r) e^{i(\theta - \omega_m t)} \right\}. \tag{5}$$

The ratios between the four modes (i.e. the coefficients A_m) are found by setting one the coefficients to be 1 and optimizing the remaining three coefficients to maximize the energy growth of the disturbance $G(t)$ by minimizing $E(0)$ (the optimal A_m is obtained analytically, $\partial E(0)/\partial A_m = 0$). From the flow equations it is possible to obtain for $n = 1$ that $\Im\{\hat{u}_{1,r}\} = \Re\{\hat{u}_{1,\theta}\} = \Im\{\hat{u}_{1,x}\} = 0$. Thus, the velocity field at $t = 0$ is given by

$$\begin{aligned}
u_{1,r}(t=0,r,\theta) &= A_0 \cos(\theta) \sum_{m=1}^{4} A_m \hat{u}_{r_m}(r), \\
u_{1,\theta}(t=0,r,\theta) &= -A_0 \sin(\theta) \sum_{m=1}^{4} A_m \hat{u}_{\theta_m}(r), \\
u_{1,x}(t=0,r,\theta) &= A_0 \cos(\theta) \sum_{m=1}^{4} A_m \hat{u}_{x_m}(r).
\end{aligned} \tag{6}$$

The radial distributions of the eigenfunctions, after summation over the four modes, are shown in Figure [2] for $A_0 = 1$.

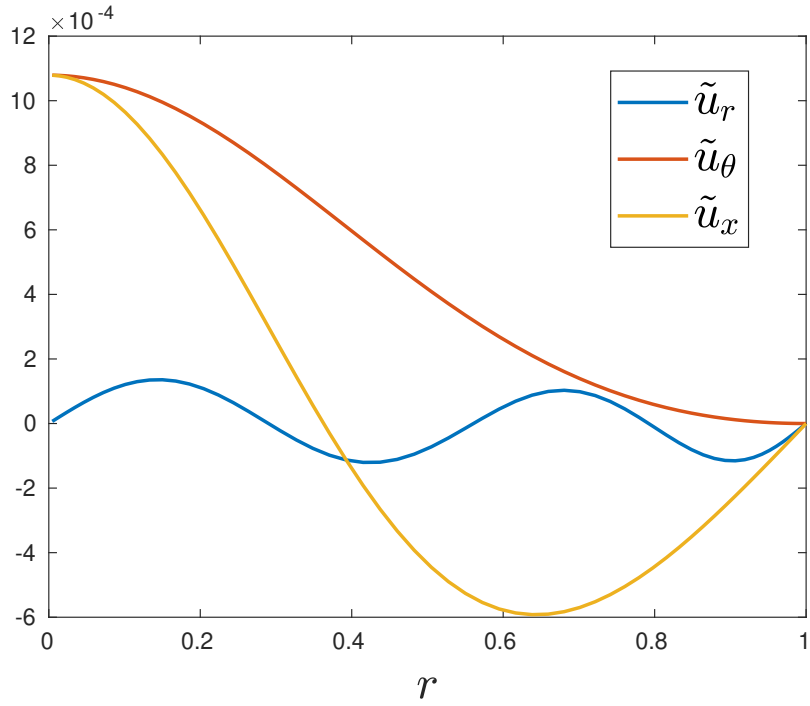


Figure 2: Initial velocities of the main disturbance (4 modes)

The initial velocity field in the cross-section is shown in Figure [3]. The initial disturbance corresponds to a CVP – a counter-rotating vortex pair.

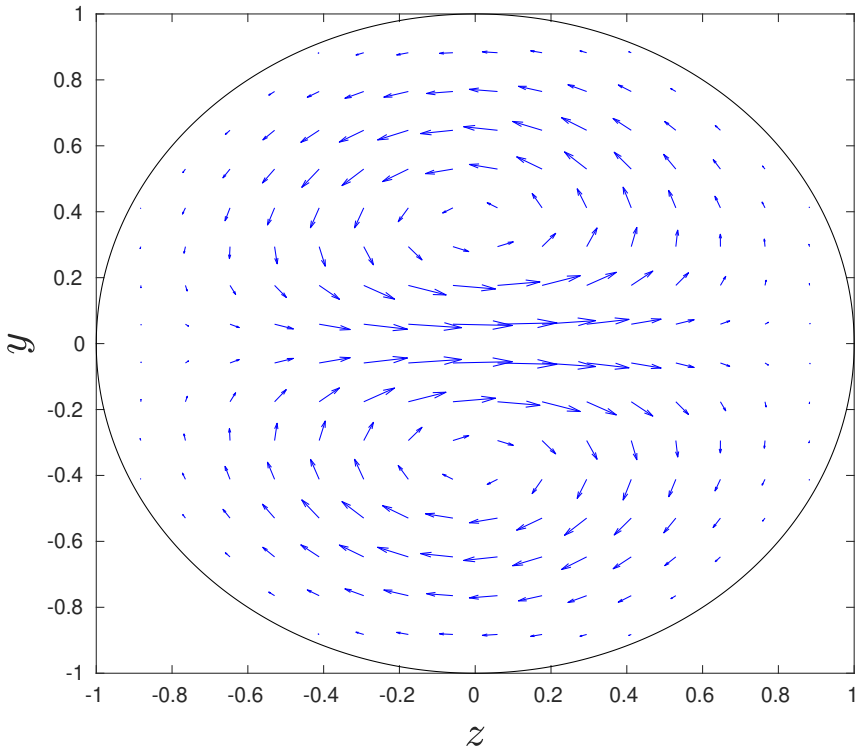


Figure 3: The initial velocity field in the cross-section

As mentioned above, all the modes of the primary disturbance eventually decay due to viscous effects. Nonetheless, we can enhance the process of TG by increasing the Reynolds number since $\omega_m \propto Re^{-1}$, or by increasing the parameter A_0.

In the next section we will introduce some of the main DNS results of part A of the project, and compare these results to an analytical approximation based on the four modes.

III. Results

The linear transient growth is traditionally estimated from the gain $G(t)$ of the disturbance kinetic energy, $E(t)$, at time t, normalized by its initial value, $E(t=0)$, where the energy is defined by the volumetric integral over the whole domain, i.e.,

$$G(t) = \frac{E(t)}{E(t=0)}, \qquad E(t) = \frac{1}{V}\oint_V |\boldsymbol{u}|^2\, dV = \frac{\alpha}{4\pi^2}\int_0^{L_x}\int_0^{2\pi}\int_0^1 (|u_r|^2 + |u_\theta|^2 + |u_x|^2)\, r\, dr d\theta dx, \tag{7}$$

where V is the volume of the pipe.
The energy growth of the optimal disturbance for pipe Poiseuille flow at a Reynolds number of $Re = 3000$ and an azimuthal wavenumber of $n = 1$ from the DNS is shown by the blue solid line in Figure [4], and is compared to the analytically predicted curve calculated using 4 least modes based on the LST (black). Note that we display the energy in the perturbation to the mean flow, namely, removing $\boldsymbol{U}_0$ from $\boldsymbol{u}_{TG}$. The good agreement between the two curves is an outcome of the low initial amplitude A_0 used in this case, which justifies neglecting the nonlinear terms. Moreover, we display the energy gain versus t/Re because the time scale of the TG is of the order $\mathcal{O}(Re)$.

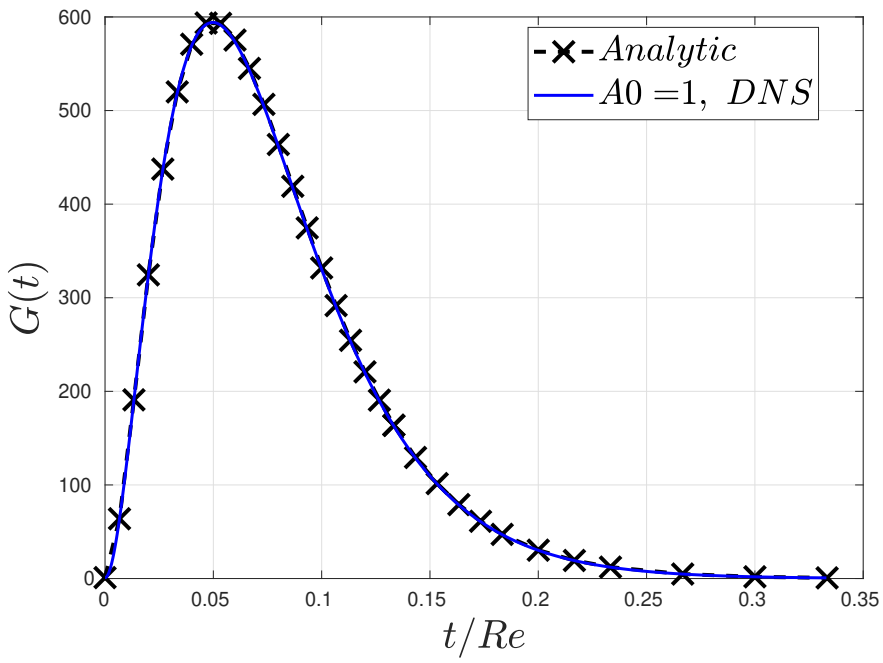


Figure 4: The energy gain of the main disturbance with $Re = 3000$. Comparison between linear analytical prediction based on four modes (black) and DNS (blue) with an initial amplitude of $A_0 = 1$

As the amplitude of the disturbance increases, the nonlinear terms become more dominant and a deviation is observed between the linear analytical prediction based on four modes (black) and the DNS (colors), as shown in Figure [5]. It is worth noting that a wrong impression may be that the energy becomes smaller as we increase the initial amplitude, A_0, however, since the kinetic energy is normalized by its initial value, the total kinetic energy increases due to the nonlinear effects.

Further validation to the model can be done by looking at the energy over a long period of time. Recall that the disturbance eventually decays exponentially over time due to viscous effects. The dominant mode after a long time is the mode that has the slowest decay rate, which is the least stable mode (the first mode). According to the stability analysis for the chosen Reynolds number, $Re = 3000$, the first mode has a decay rate ($\omega_i < 0$) of $\omega_{1,i} = -0.00489$. From equations (3) and (7) it can be seen that for long times both the kinetic energy and the energy gain are proportional to $\exp(2\omega_{1,i}t)$. This trend is observed in the DNS by examining the energy gain on a y-log scale for several initial amplitudes A_0 (colors), shown in Figure [6]. The predicted decay is superimposed on the figure by a dashed black line. Again, there is good agreement between the DNS results and the analytical prediction.

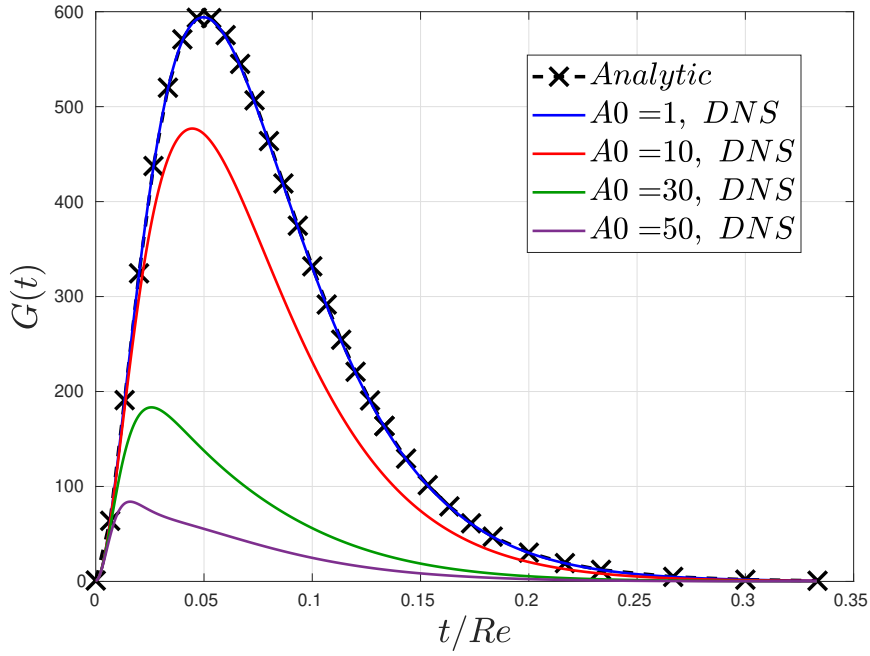


Figure 5: The energy gain of the main disturbance with $Re = 3000$. Comparison of the DNS results with several initial amplitudes A_0 (colored curves)

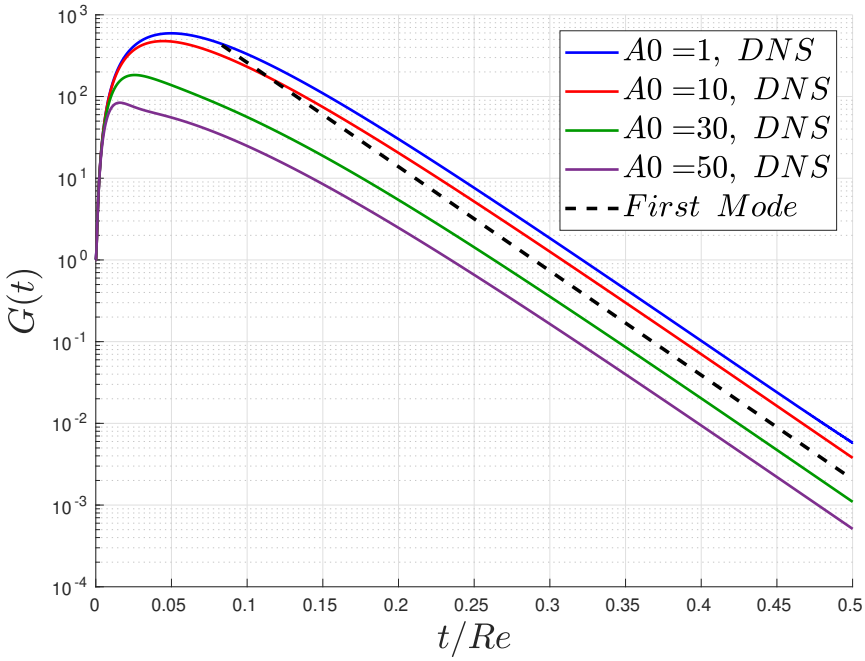


Figure 6: The energy gain of the main disturbance from the DNS of several amplitudes compared to the predicted slope of the first mode for $Re = 3000$

Let us now turn our attention to the velocities of the disturbance (removing the base flow) and examine the effects of the initial amplitude A_0 at several times. The velocity field during the transient growth process remains streamwise independent, i.e. its axial wavenumber is $\alpha = 0$, however, in general it may consist of many azimuthal wavenumbers. The initial disturbance has an azimuthal wavenumber $n = 1$, nevertheless, in the nonlinear case a mean flow modification, $n = 0$, may appear, along with the harmonic $n = 2$, as well as higher wavenumbers. Using Fourier modal decomposition we can examine the velocity components at different azimuthal wavenumbers, n, to better understand the nonlinear effects observed in the graphs of the energy.

The velocity profiles $\hat{u}_{n,x}$, $\hat{u}_{n,r}$ and $\hat{u}_{n,\theta}$, where n equals either 0 (green), 1 (blue) or 2 (red) are shown in Figures [7-10]. We can in see Figures [7] and [9] that at time $t = 0$ the disturbance consists of $n = 1$ only. After a certain time, due to nonlinear effects, the $n = 1$ mode can interact with itself and excite other modes. For an order of $\mathcal{O}(\varepsilon^2)$ the the excited modes are those having $n = 0$ and $n = 2$. For the case with an initial amplitude of $A_0 = 1$ there are almost no velocities corresponding to $n \neq 1$ as can be seen in Figure [8]. Nevertheless, for higher amplitudes, such as $A_0 = 10$, nonlinear terms become significant as shown in Figure [10] (see also Figure [5]). It can be seen that the analytical prediction tracks the velocities profiles quite well provided the nonlinear effects are small.

In addition, from the pair of Figures [7-8] and [9-10], it can be seen that the $\hat{u}_r$ and $\hat{u}_\theta$ velocity profiles decay in time since the CVP decays, while the growth occurs in the axial component $\hat{u}_x$ due to lift-up. Moreover, the shapes of the velocity profiles of $\hat{u}_r$ and $\hat{u}_\theta$ hardly change whereas the shape of the velocity profiles in the axial direction $\hat{u}_x$ evolves and changes appreciably with time.

IV. Conclusions

In this project it is shown that the transient growth scenario in pipe Poiseuille flow can be represented by only four linearly decaying modes, given analytically. Comparison of the DNS results to the analytical expressions showed that for small initial amplitudes there is a perfect agreement between the two. Nevertheless, as the amplitudes are increased, it becomes necessary to consider the nonlinear effects in the calculations.

Part B of this project discusses the secondary stability properties of the modified base flow, including the base flow and the primary disturbance undergoing transient growth. With the understanding of the TG mechanism in basic flows such as Couette flow, plane Poiseuille flow and pipe Poiseuille flow, we can better understand complex instability mechanisms associated with more complex flows, e.g. boundary layers.

Acknowledgments

The author would like to express his very great appreciation to Dr. Karp and to Prof. Cohen for their valuable and constructive suggestions during the planning and development of this research work. Their willingness to give their time so generously is very much appreciated.

Disclaimers

#1: This report is the first of 2 parts, dealing with transition to turbulence in a pipe. The second report is entitled 'Transition to turbulence in a pipe – part B – Simulation of exponential growth of secondary perturbations'. The division into 2 parts was made for convenience.

#2: The research is part of a broad and comprehensive work on transition to turbulence in a pipe led by Prof. Jacob Cohen and Dr. Michael Karp. The work was carried out in collaboration with Or Nataf. The main goal of the project was to run the direct numerical simulation (DNS), while the analytical model and stability analysis as well as most of the Matlab code files were provided by Prof. Cohen and Dr. Karp, who had previously studied transition to turbulence in Couette flow and in plane Poiseuille flow.

References

[1] M. Karp and J. Cohen. "Tracking stages of transition in Couette flow analytically". *Journal of fluid mechanics* 748 (2014), pp. 896–931.

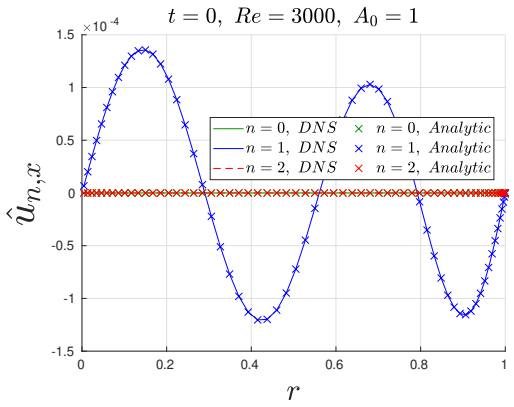
[2] F. Roizner, M. Karp, and J. Cohen. "Subcritical transition in plane Poiseuille flow as a linear instability process". *Physics of Fluids* 28.5 (2016), p. 054104.

[3] A. P. Willis. "The Openpipeflow Navier–Stokes Solver". *SoftwareX* 6 (2017), pp. 124–127. DOI: https://doi.org/10.1016/j.softx.2017.05.003. URL: http://arxiv.org/abs/1705.03838.

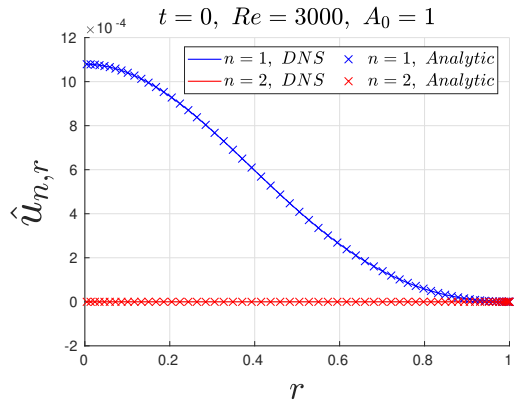
[4] M. Karp, F. Roizner, and J. Cohen. "Analytical Model for Transition in Couette and Poiseuille Flows". *Procedia IUTAM* 14 (2015), pp. 227–235.

[5] M. Karp and J. Cohen. "Transition to turbulence in Couette flow". *The 53rd Israel Annual Conference on Aerospace Sciences* (2013).

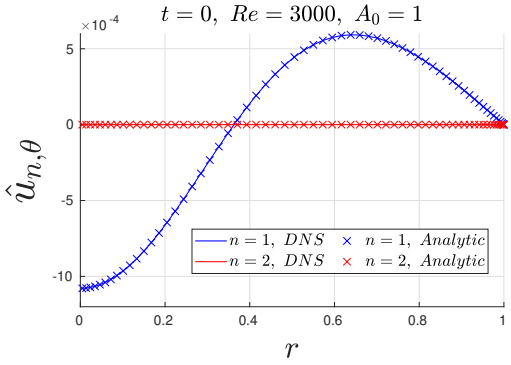
[6] F. Roizner, M. Karp, and J. Cohen. "Transition to turbulence in plane Poiseuille flow". *The 55th Israel Annual Conference on Aerospace Sciences* (2015).



(a) $\hat{u}_{n,x}$

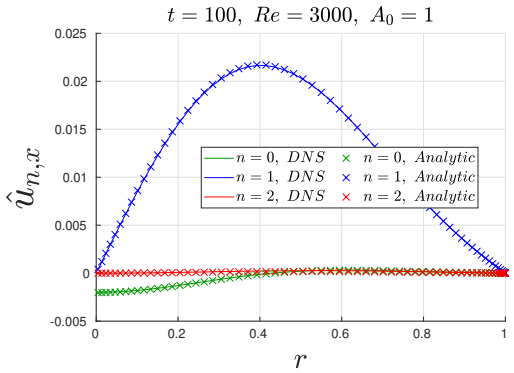


(b) $\hat{u}_{n,r}$

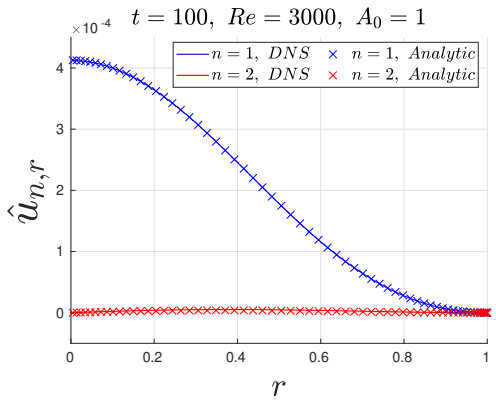


(c) $\hat{u}_{n,\theta}$

Figure 7: Velocity components in Fourier space $(\hat{u}_{n,x}, \hat{u}_{n,r}, \hat{u}_{n,\theta})$ for $A_0 = 1$, $Re = 3000$ at $t = 0$



(a) $\hat{u}_{n,x}$



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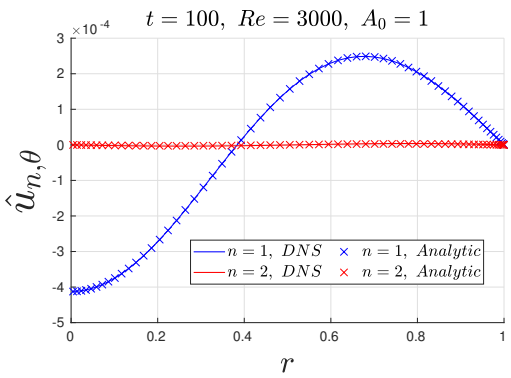
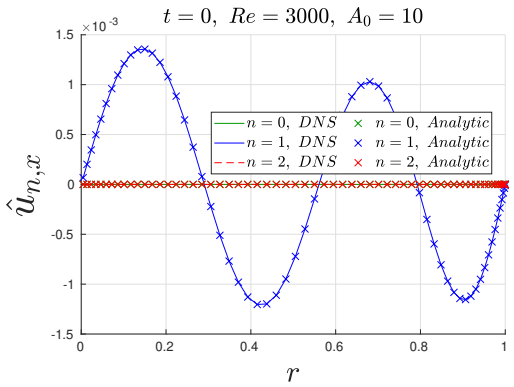
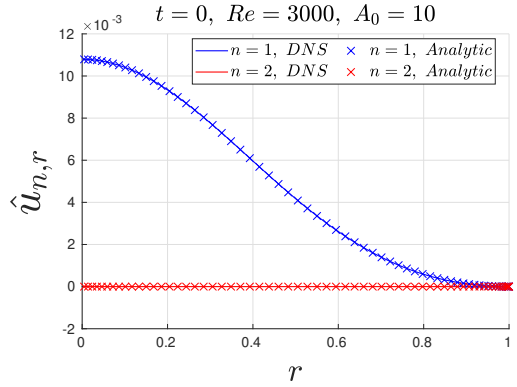


(c) $\hat{u}_{n,\theta}$

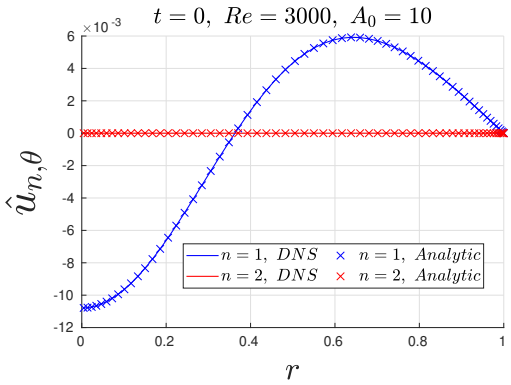
Figure 8: Velocity components in Fourier space $(\hat{u}_{n,x}, \hat{u}_{n,r}, \hat{u}_{n,\theta})$ for $A_0 = 1$, $Re = 3000$ at $t = 100$



(a) $\hat{u}_{n,x}$

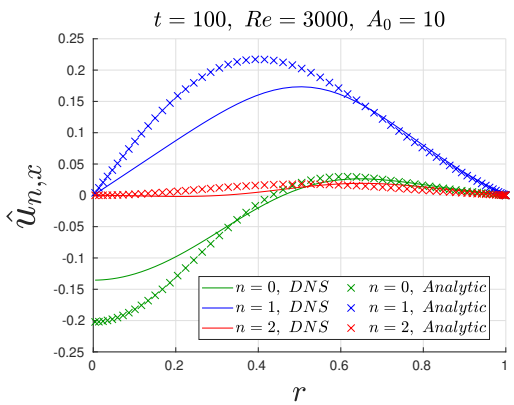


(b) $\hat{u}_{n,r}$

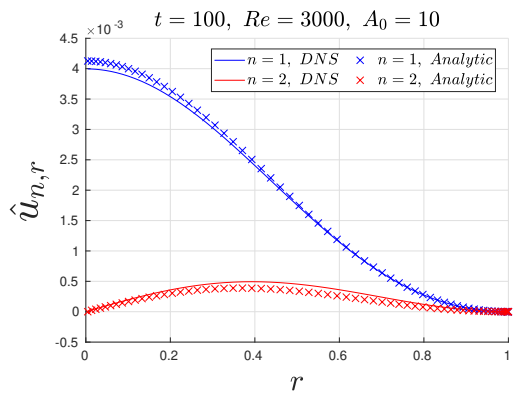


(c) $\hat{u}_{n,\theta}$

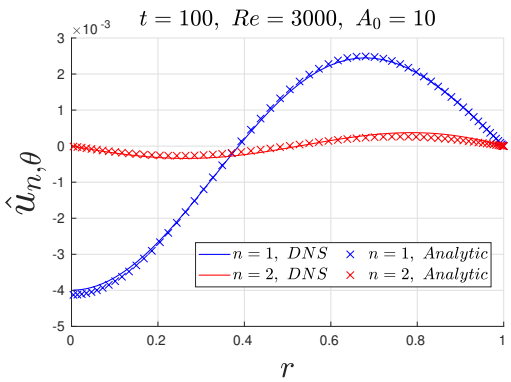
Figure 9: Velocity components in Fourier space $(\hat{u}_{n,x}, \hat{u}_{n,r}, \hat{u}_{n,\theta})$ for $A_0 = 10$, $Re = 3000$ at $t = 0$



(a) $\hat{u}_{n,x}$



(b) $\hat{u}_{n,r}$



(c) $\hat{u}_{n,\theta}$

Figure 10: Velocity components in Fourier space $(\hat{u}_{n,x}, \hat{u}_{n,r}, \hat{u}_{n,\theta})$ for $A_0 = 10$, $Re = 3000$ at $t = 100$